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Real Functions in Several Variables: Volume IX

sformation of Integrals and Improper Integrals

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Contents


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thinking.

Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral.*

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation.* However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume IX, Transformation formulæ and improper integrals

This is the ninth volume in the series of books on *Real Functions in Several Variables.*

In Chapter 29 we investigate how to change variables in the integrals in the plane or space. It is shown that the previous chapters are special cases of this general theory. In particular, we obtain a new introduction of the various weight functions. Formally, Chapter 29 would suffice for the theory, but we have chosen for pedagogical reasons to describe separately the plane integral in rectangular or polar coordinates, the space integral in rectangular, semi-polar or spherical coordinates, the line integral and the surface integral, because then we can identify the weight function, which should be used in each case.

In Chapter 30 we look at the cases, where $f : A \to \mathbb{R}$ is continuous, but A is either unbounded or not closed. In this case the integrand may tend to $\pm\infty$, when $\mathbf{x}$ approaches a boundary point. Such integrals called improper integrals.

One should for improper integrals always split the integrand f into its positive and negative parts, i.e.

$$f(\mathbf{x}) = f_+(\mathbf{x}) - f_-(\mathbf{x}),$$

where

$$f_+(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) > 0, \\ 0 & \text{if } f(\mathbf{x}) \le 0, \end{cases} \qquad f_-(\mathbf{x}) := \begin{cases} 0 & \text{if } f(\mathbf{x}) > 0, \\ -f(\mathbf{x}) & \text{if } f(\mathbf{x}) \le 0, \end{cases}$$

where for technical reasons both f_+ and f_- are nonnegative. Then we discuss the possible definition of

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu, \qquad \text{provided that } f(\mathbf{x}) \ge 0 \text{ for all } \mathbf{x} \in A,$$

i.e. for f nonnegative. Or, alternatively, split the domain $A = A_+ \cup A_-$, where $f > 0$ on A_+ and $f \le 0$ on A_-.
If both f_+ and f_- have finite values of their (proper or improper) integrals, then we say that

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu := \int_A f_+(\mathbf{x})\,\mathrm{d}\mu - \int_A f_-(\mathbf{x})\,\mathrm{d}\mu = \int_{A_+} f(\mathbf{x})\,\mathrm{d}\mu + \int_{A_-} f(\mathbf{x})\,\mathrm{d}\mu \quad (\in \mathbb{R}),$$

is convergent.

All the integrals above, plane, space and surface integrals, can be improper. To decide whether they are convergent or not, we split them as above and then use a truncation technique and finally let the truncations shrink towards the "singular points" on the (intrinsic) boundary of A to see if we have convergence or divergence.

29 Transformation of plane and space integrals

29.1 Transformation of a plane integral

We shall in this section see how we can integrate a plane integral by using a change of variables.

Consider two bounded and closed plane sets B, $D \subset \mathbb{R}^2$, and let

$$\mathbf{r} = (r_1, r_2) : D \to \mathbb{R}^2$$

be a C^1 vector function, which satisfies

1) The vector function $\mathbf{r}$ maps D onto B, i.e. $\mathbf{r}(D) = B$, so it is surjective.

2) The vector function $\mathbf{r}$ is injective almost everywhere.

We use the coordinates $(x, y) \in B$ and $(u, v) \in D$, so we have

$$x = r_1(u, v) \qquad \text{and} \qquad y = r_2(u, v).$$

If we consider B as a *surface*, and not just a plane set, then $\int_B f(x, y)\, \mathrm{d}S$ can be viewed as a *surface interal*, so we get from Chapter 27 that the reduction formula is

$$\int_B f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_D f\left(r_1(u, v), r_2(u, v)\right) \cdot \|\mathbf{N}(u, v)\|\, \mathrm{d}u\, \mathrm{d}v.$$

When the plane domain (surface) B is imbedded in $\mathbb{R}^3$, we can use the following rectangular description,

$$B = \{(x, y, 0) \mid x = r_1(u, v) \text{ and } y = r_2(u, v) \text{ for } (u, v) \in D\}.$$

Then the *normal vector* $\mathbf{N}$ is parallel with the z-axis, and we get

$$\mathbf{N}(u, v) = \det \begin{pmatrix} \dfrac{\partial r_1}{\partial u} & \dfrac{\partial r_1}{\partial v} & 0 \\ \dfrac{\partial r_2}{\partial u} & \dfrac{\partial r_2}{\partial v} & 0 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = \det \begin{pmatrix} \dfrac{\partial r_1}{\partial u} & \dfrac{\partial r_1}{\partial v} \\ \dfrac{\partial r_2}{\partial u} & \dfrac{\partial r_2}{\partial v} \end{pmatrix} \mathbf{e}_3,$$

so we have calculated the weight function, which is the absolute value of the so-called *Jacobian*, defined by

$$\frac{\partial(x, y)}{\partial(u, v)} := \det \begin{pmatrix} \dfrac{\partial r_1}{\partial u} & \dfrac{\partial r_1}{\partial v} \\ \dfrac{\partial r_2}{\partial u} & \dfrac{\partial r_2}{\partial v} \end{pmatrix}, \qquad \|\mathbf{N}(u, v)\| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|.$$

The requirement from Chapter 27 is that the weight function is $\neq 0$ almost everywhere. We can therefore formulate (and again without a correct proof) the following theorem,

Theorem 29.1 The transformation theorem for a plane integral. *Let $(x, y) \in B$ and $(u, v) \in D$, where B and D are bounded and closed sets in the (x, y)-plane, the (u, v)-plane resp.. Assume that*

$$\mathbf{r} = (r_1, r_2) : D \to \mathbb{R}^2$$

is a C^1 vector function, such that

1) The vector function maps D onto B, i.e. $\mathbf{r}(D) = B$, and $x = r_1(u, v)$ and $y = r_2(u, v)$.

2) The vector function $\mathbf{r}$ is injective almost everywhere in D.

3) The Jacobian is $\neq 0$ almost everywhere, i.e.

$$\frac{\partial(x, y)}{\partial(u, v)} := \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \neq 0 \qquad \textit{almost everywhere.}$$

If $f; B \to \mathbb{R}$ is a continuous function, then we have the reduction formula

$$\int_B f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_D f\left(r_1(u, v), r_2(u, v)\right) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\, \mathrm{d}v.$$

It is for mnemotechnical reasons that we use the old-fashioned notation of the *Jacobian*. The reason is that we then remember that intuitively $\mathrm{d}u\,\mathrm{d}v$ in the "numerator" is "cancelled" by the symbol "$\partial(u,v)$" in the "denominator", leaving "$\partial(x,y)$" in the "numerator", which is more or less the same as $\mathrm{d}x\,\mathrm{d}y$ on the left hand side of the transformation formula. This incorrect notation reminds us that the *Jacobian* is a function of (u,v), which is more difficult to derive, when we use a more correct notation.

As a simple check, let us consider the change from rectangular coordinate in the plane to polar coordinates, so $\mathbf{r}$ is given by

$$x = \varrho\cos\varphi \qquad \text{and} \qquad y = \varrho\sin\varphi.$$

Then

$$\frac{\partial(x,y)}{\partial(\varrho,\varphi)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial \varrho} & \dfrac{\partial x}{\partial \varphi} \\[2mm] \dfrac{\partial y}{\partial \varrho} & \dfrac{\partial y}{\partial \varphi}\end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -\varrho\sin\varphi \\ \sin\varphi & \varrho\cos\varphi\end{pmatrix} = \varrho,$$

which is precisely the weight function we found previously in Chapter 20, when we used polar coordinates, so we have again derived the well-known formula

$$\int_B f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_D f(\varrho\cos\varphi,\varrho\sin\varphi)\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi.$$

29.2 Transformation of a space integral

Since in the previous section

$$\|\mathbf{N}(u,v)\| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}\right|$$

is the area of the parallelogram formed by the tangents of the two parameter curves, this gives us a clue of what we should expect in three
The analogue in three dimensions is a map $\mathbf{r} = (r_1,r_2,r_3) : D \to B$, where we use the notation

$$(x,y,z) = (r_1(u,v,w), r_2(u,v,w), r_3(u,v,w)),$$

and where we assume that $\mathbf{r}$ is surjective, and injective almost everywhere.

The *Jacobian* is here

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{pmatrix}.$$

It is well-known from *Linear Algebra* that the absolute value of the *Jacobian* is the volume of the parallelepipidum spanned by the three tangents of the three parameter curves at a given point. In other words, our parallelepipeda are the building stones, which we use when we build up the 3-dimensional integral, and their volumes, the absolute value of the Jacobian, form the weight function.

The above makes the following theorem plausible. We quote it – as usual without a correct proof.

Theorem 29.2 Transformation theorem for a space integral *Let* $(x, y, z) \in B$ *and* $(u, v, w) \in D$, *where* $B \subset \mathbb{R}^3$ *and* $D \subset \mathbb{R}^3$ *are bounded and closed sets. Assume that*

$$\mathbf{r} = (r_1, r_2, r_3) : D \to \mathbb{R}^3$$

is a C^1 *vector function, such that*

1) The vector function $\mathbf{r}$ *maps* D *onto* B, *i.e.* $\mathbf{r}(D) = B$, *and*

$$x = r_1(u, v, w), \qquad y = r_2(u, v, w), \qquad z = r_3(u, v, w).$$

2) The function $\mathbf{r}$ *is injective almost everywhere.*

3) The Jacobian is $\neq 0$ *almost everywhere, i.e.*

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{pmatrix} \neq 0 \qquad \textit{almost everywhere.}$$

If $f : B \to \mathbb{R}$ *is a continuous function, then the reduction formula is*

$$\int_B f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_D f\left(r_1(u, v, w), r_2(u, v, w), r_3(u, v, w)\right) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Let us check this formula by transforming the integral in rectangular coordinates into spherical coordinates, i.e. $\mathbf{r}$ is here specified by

$$x = r \sin\theta \cos\varphi, \qquad y = r \sin\theta \cos\varphi, \qquad z = r \cos\theta.$$

Then the Jacobian is

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} &= \det \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\cos\varphi & r\cos\theta\sin\varphi & r\cos\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \\ &= \cos\theta \det \begin{pmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} + r\sin\theta \det \begin{pmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} \\ &= r^2\cos^2\theta\sin\theta \det \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} + r^2\sin^3\theta \det \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \\ &= r^2\sin\theta\left\{\cos^2\theta + \sin^2\theta\right\} = r^2\sin\theta, \end{aligned}$$

so we have indeed obtained the same weight function as we found in Chapter 24.

29.3 Procedures for the transformation of plane or space integrals

All the reduction formulæ in the previous chapters are special cases of more general formulæ. The presentations were using the classical coordinate systems: the *rectangular, polar, semi-polar, and the spherical* coordinate systems. When the coordinate system under consideration is *not* one of these we must use the general formulæ form the present section instead.

A. Dimension 2.

1) Find a suitable parameter representation $(x, y) = \mathbf{r}(u, v)$, $(u, v) \in D$.
Sketch the parametric domain D and argue *briefly* that $\mathbf{r}(u, v)$ is injective, with the exception of e.g. a *finite* number of points. (More precisely one can neglect a so-called null set; which usually is not defined in elementary courses in Calculus).
Show also that the *range* is $\mathbf{r}(D) = B$.

2) Calculate the *Jacobian*

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

Note here that one is *forced* to find x and y as *functions of* (u, v) in 1) in order to calculate the Jacobian.

The *area element* is

$$\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v,$$

where the area element $\mathrm{d}x\,\mathrm{d}y$ is lying in B, while the area element $\mathrm{d}u\,\mathrm{d}v$ is lying in the parametric domain D.

3) Insert the result and calculate the right hand side by previous known methods in the expression

$$\int_B f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_D f(\mathbf{r}(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

Remark 29.1 Usually u and v are given as functions in x and y instead of the form we shall use:

(29.1) $u = U(x, y)$ and $v = V(x, y)$.

Then one has to solve these equations with respect to x and y. If we in this way obtain a unique solution, then we have at the same time implicitly proved that the map is injective. Apply furthermore (29.1) to find the images of the boundary curves of B, thereby finding the boundary of D. Finally, the parametric domain D is identified. $\diamond$

B. Dimension 3.

Formally the procedure is the same as in section A with obvious modifications due to the higher dimension.

1) Find a suitable parametric representation $(x,y,z) = \mathbf{r}(u,v,w)$, $(u,v,w) \in D$. (This will usually be given, possibly in the form $u = U(x,y,z)$, $v = V(x,y,z)$, $w = W(x,y,z)$. If so, solve these equations with respect to x, y, z).
Sketch the parametric domain D and argue (briefly) that the mapping $\mathbf{r}(u,v,w)$ is injective almost everywhere.
Show that the *range* is $\mathbf{r}(D) = B$. Cf. also the remarks to section A.
2) Calculate the **Jacobian**

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Then the *volume element* is

$$\mathrm{d}\Omega = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w,$$

where one must be careful not to forget the numerical signs of the Jacobian.
3) Insert and calculate the right hand side by means of one of the previous methods in the formula

$$\int_B f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_D f(\mathbf{r}(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

29.4 Examples of transformation of plane and space integrals

Example 29.1

A. *Calculate the plane integral*

$$I = \int_B \cos\left(\frac{y-x}{y+x}\right) \mathrm{d}x\,\mathrm{d}y$$

over the trapeze shown on the figure.

D. A direct calculation applying one of the usual reduction theorems is not possible, because none of the forms

$$\int \cos\left(\frac{y-x}{y+x}\right) \mathrm{d}x = \int \cos\left(\frac{2y}{y+x} - 1\right) \mathrm{d}x = \int \cos\left(1 - \frac{2x}{y+x}\right) \mathrm{d}y$$

can be integrated within the realm of our known functions. The situation is even worse in polar coordinates. Therefore, the only possibility left is to find a convenient transform, such that the integrand becomes more easy to handle.

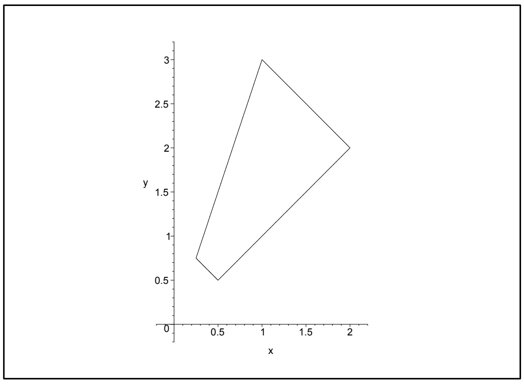

Figure 29.1: The trapeze B.

The unpleasant thing is of course the fraction $\dfrac{y-x}{y+x}$. One idea would be to introduce the numerator as a new variable, and the denominator as another new variable. If we do this, then we must show that we obtain a *unique* correspondence between the domain B and a *parametric domain* D, which also should be found. Finally we shall find the *Jacobian*. When we have found all the terms in the transformation formula, then calculate the integral.

Remark 29.2 This time we see that it is here quite helpful to start the discussion in **D**, which is not common knowledge from high school. First we discuss the problem. Based on this discussion we make a decision on the further procedure. $\Diamond$

I. According to **D**. we choose the numerator and the denominator as our new variables. Most people would here choose the *numerator* as u and the *denominator* as v, so we shall do the same, although it can be shown that we shall get simpler calculations if we interchange the definition of u and v.

We therefore put as the most natural choice

(29.2) numerator : $\quad u = y - x \quad$ and $\quad$ denominator : $\quad v = y + x.$

Then we shall prove that this gives a *one-to-one* correspondence. This means that we for any given u and v obtain unique solutions x and y:

$$x = \frac{v-y}{2} \qquad \text{and} \qquad y = \frac{u+v}{2}.$$

Obviously the transform is *continuous* both ways. Since B is *closed and bounded*, the range D by this transform is again *closed and bounded*, cf. the important *second main theorem for continuous functions*.

Since the transform is one-to-one *everywhere*, the boundary ∂B is mapped one-to-one onto the boundary ∂D. This is expressed in the following way:

1) The line $x + y = 1$ corresponds by (29.2) to $v = 1$.
2) The line $y = x$, i.e. $y - x = 0$, corresponds by (29.2) to $u = 0$.
3) The line $y + x = 4$ corresponds by (29.2) to $v = 4$.
4) The line $y = 3x$ corresponds to $\dfrac{u+v}{2} = 3\,\dfrac{v-u}{2}$, i.e. to $v = 2u$.

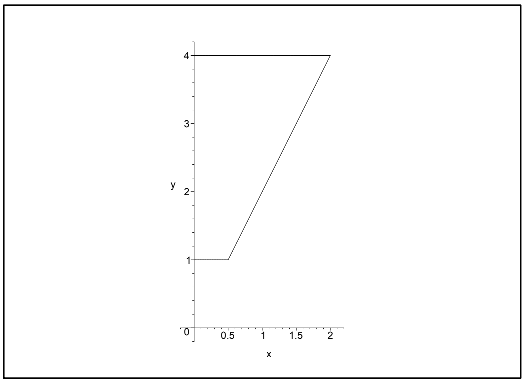

Figure 29.2: The parametric domain D. The oblique line has the equation $v = 2u$ or $u = \dfrac{1}{2}\,v$ as its representation.

The only *closed and bounded* domain in the (u, v)-plane, which has the new boundary curves as its boundary is D as indicated on the figure. In practice one draws the lines $v = 1$, $u = 0$, $v = 4$ and

$v = 2u$ and use the figure to find out where the bounded set D is situated, such that the boundary consists of parts of all four lines.

Then we calculate the *weight function* $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

The "old fashioned" notation above indicates that we shall use the transform, where x and y (in the "numerator") are functions of u and v (in the "denominator"), i.e.

$$x = \frac{1}{2}\,v - \frac{1}{2}\,u \qquad \text{and} \qquad y = \frac{1}{2}\,u + \frac{1}{2}\,v.$$

This gives us the *Jacobian*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\[2mm] \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}\cdot\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = \frac{1}{4}\cdot(-2) = -\frac{1}{2}.$$

It follows that the Jacobian is *negative*, hence the *weight function* becomes

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

Remark 29.3 If we here interchange u and v in (29.2), then we obtain that the Jacobian becomes *positive*. $\diamondsuit$

We have now come to the *reduction formula*,

$$I = \int_B \cos\left(\frac{y-x}{y+x}\right)\,\mathrm{d}x\,\mathrm{d}y = \int_D \cos\left(\frac{u}{v}\right)\cdot\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v.$$

Note that both sides here are *abstract plane integrals*.

We see on the right hand side that $\int \cos\left(\dfrac{u}{v}\right)\,dv$ cannot be integrated within the realm of our known arsenal of functions. But $\int \cos\left(\dfrac{u}{v}\right)\,du$ can! Therefore, when we reduce the plane integral on the right hand side we put the u-integral as the inner integral. Then

$$I = \frac{1}{2}\int_D \cos\left(\frac{u}{v}\right)\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_1^4 \left\{\int_0^{\frac{v}{2}} \cos\left(\frac{u}{v}\right)\,\mathrm{d}u\right\}\,\mathrm{d}v.$$

When $v \neq 0$ is kept constant, we get from the inner integral

$$\int_0^{\frac{v}{2}} \cos\left(\frac{u}{v}\right)\,\mathrm{d}u = \left[v\,\sin\left(\frac{u}{v}\right)\right]_0^{\frac{v}{2}} = v\,\sin\left(\frac{v}{2}\cdot\frac{1}{v}\right) = \sin\left(\frac{1}{2}\right)\cdot v,$$

where $\sin\left(\dfrac{1}{2}\right)$ is a constant, which shall *not* be found explicitly! (Note that at $\dfrac{1}{2}$ radian is *not* equal to $\dfrac{\pi}{2}$).

Finally we get by insertion

$$I = \frac{1}{2} \int_1^4 \sin\left(\frac{1}{2}\right) \cdot v \, dv = \frac{1}{2} \sin\left(\frac{1}{2}\right) \cdot \left[\frac{1}{2} v^2\right]_1^4 = \frac{15}{4} \sin\left(\frac{1}{2}\right). \qquad \Diamond$$

Example 29.2

A. *Let*

$$A = \{(x, y, z) \mid 0 \le x \le 1,\, 0 \le y \le \sqrt{x},\, -y \le z \le y\}.$$

calculate

$$I = \int_A \frac{\exp\left((2-y-z)^3\right)}{4+y+z}\,\mathrm{d}\Omega.$$

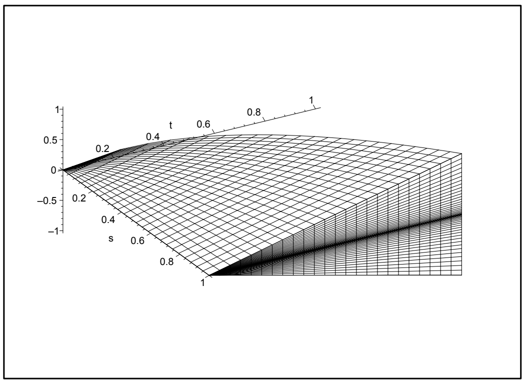

Figure 29.3: The domain A. Note the different scales on the axes.

D. Let us start by pulling out the teeth of this big and horrible example! Its purpose is *only* to demonstrate that even apparent incalculable integrals in some cases nevertheless can be calculated by using a "convenient transform". This example is from a textbook, where earlier students got the wrong impression that "every application of the transformation theorem looks like this example", which is not true. Without this extra comment this example will send a *wrong* message to the reader.

Let us first discuss, how we can find a reasonable transform. I shall follow more or less the way of thinking which the author of this example must have used, the first time it was created.

At the end of this example I shall describe the very modest requirements which may be demanded of the students. In other words, this example should only be used as an *inspiration* for other similar problems which may occur in practice.

I. Let us start by looking at the *geometry* of A. The projection B of A onto the (x, y)-plane is

$$B = \{(x, y) \mid 0 \le x \le 1,\, 0 \le y \le \sqrt{x}\}.$$

Since A for every $x \in\,]0, 1]$ is cut into an isosceles rectangular triangle

$$\Delta_x = \{(y, z) \mid 0 \le y \le \sqrt{x},\, -y \le z \le y\},$$

it is easy to sketch A, cf. a previous figure.

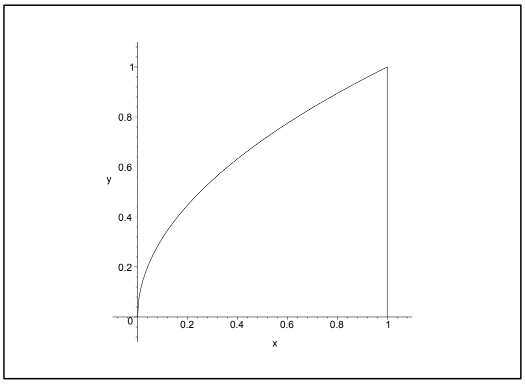

Figure 29.4: The projection B of the domain A onto the (x, y)-plane.

Then the integrand

$$\frac{\exp\left((2-y-z)^3\right)}{4+y+z}$$

should be "straightened out". A reasonable guess would be to introduce

$$u = y + z.$$

Remark 29.4 Once we have gone through all the calculations it can be seen that

$$\tilde{u} = \frac{y+z}{2},$$

would be a better choice, because then we would get rid of a lot of irritating constants. Of pedagogical reasons we shall *not* here use the most optimal transform, but instead the transform which one would expect the student to choose. $\Diamond$

Since we do not get further information from the integrand, we shall turn to the domain A. The boundary of A is (almost) determined by putting equality sign into the definition of A instead of $\leq$. First everything is written in a "binary" way in the definition of A,

$$\begin{aligned} A &= \{(x,y,z) \mid 0 \leq x \leq 1 \,\wedge\, 0 \leq y \leq \sqrt{x} \,\wedge\, -y \leq z \leq y\} \\ &= \{(x,y,z) \mid 0 \leq x \,\wedge\, x \leq 1 \,\wedge\, 0 \leq y \,\wedge\, 0 \leq \sqrt{x} - y \,\wedge\, 0 \leq y+z \,\wedge\, 0 \leq y-z\}, \end{aligned}$$

i.e. every condition which is defining A contains only one inequality sign and one of the sides of the inequality is a constant.

We see that there are composed expressions in the latter three conditions,

$$\sqrt{x} - y \geq 0, \qquad y + z \geq 0, \qquad y - z \geq 0.$$

Since we already have chosen $u = y + z \geq 0$, we get the inspiration of choosing the new variables

$$(29.3)\quad u = y + z \geq 0, \qquad v = y - z \geq 0, \qquad w = \sqrt{x} - y \geq 0,$$

where we have taken the most ugly term, $\sqrt{x} - y$ and put it equal to w, i.e.

$$w = \sqrt{x} - y.$$

We note that we by these choices have obtained that u, v, $w \geq 0$, and that equality signs must correspond to boundary points in the (u, v, w)-space for the parametric domain D.

Next we show that the transform (29.3) is one-to-one. i.e. we shall express x, y, z *uniquely* by u, v, w. We get immediately from the first two equations that

$$y = \frac{u+v}{2} \qquad \text{and} \qquad z = \frac{u-v}{2}.$$

From the third equation we get

$$\sqrt{x} = w + y = w + \frac{u+v}{2} = \frac{1}{2}(u + v + 2w),$$

which obviously is ≥ 0, because u, v, $w \geq 0$. Therefore, by squaring,

$$x = \frac{1}{4}(u + v + 2w)^2.$$

Thus, x, y, z are uniquely determined by u, v, w, so the transform is *one-to-one*.

Since the transform and its inverse are both *continuous* and the domain A is *closed and bounded*, it follows from the *second main theorem for continuous functions* that D is also *closed and bounded*. It follows from the binary representation of A that ∂A is a subset of the union of the surfaces $x = 0$, $x = 1$, $y = 0$, $\sqrt{x} - y = 0$, $y + z <= 0$ and $y - z = 0$. These are now investigated one by one.

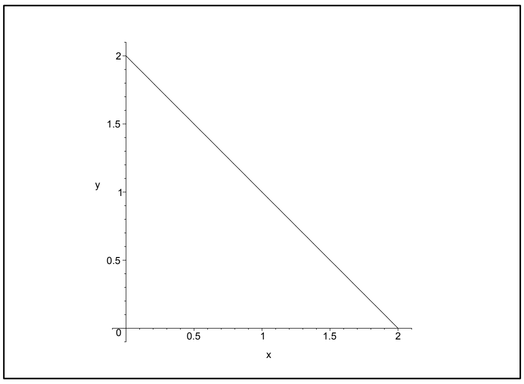

Figure 29.5: The projection of the parametric domain D in the (u, v)-plane.

1) The plane $x = 0$ corresponds to $\frac{1}{4}(u + v + 2w)^2 = 0$. Since u, v, $w \geq 0$, we only get $(u, v, w) = (0, 0, 0)$, which is in agreement with the figure of A, because the plane $x = 0$ just cuts A in $\mathbf{0}$.

2) The plane $x = 2$ corresponds to $\frac{1}{4}(u+v+w)^2 = 1$, i.e. $\frac{1}{2}(u+v+2w) = +1$, from which

$$w = 1 - \frac{u+v}{2} \geq 0.$$

Here we have again used that u, v, $w \geq 0$. Note that we also get that

$$u + v \leq 2.$$

3) The plane $y = 0$ corresponds to $\frac{1}{2}(u+v) = 0$, i.e. $u + v = 0$.

4) The remaining conditions have been found previously for $u = 0$, $v = 0$ and $w = 0$.

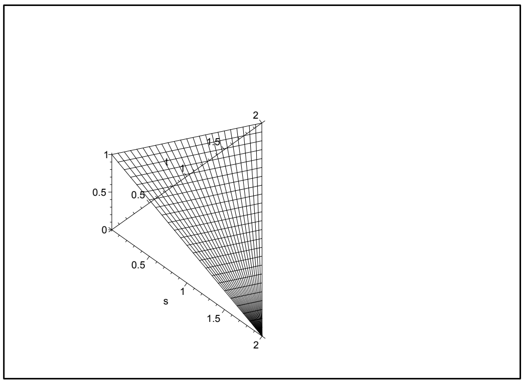

Figure 29.6: The parametric domain D.

Summing up we find that the *parametric domain* is given by

$$\begin{aligned} D &= \left\{(u,v,w) \;\middle|\; 0 \le u,\, 0 \le u+v \le 2,\, 0 \le w \le 1 - \frac{u+v}{2}\right\} \\ &= \left\{(u,v,w) \;\middle|\; 0 \le u \le 2,\, 0 \le v \le 2-u,\, 0 \le w \le 1 - \frac{u+v}{2}\right\} \\ &= \left\{(u,v,w) \;\middle|\; (u,v) \in B,\, 0 \le w \le 1 - \frac{u+v}{2}\right\}, \end{aligned}$$

where the projection B in the (u,v)-plane is given by

$$B = \{(u,v) \mid 0 \le u \le 2,\, 0 \le v \le 2-u\},$$

so B and D are now easily sketched.

By the chosen transform the *integrand* is carried over into

$$\frac{\exp\left((2-y-z)^3\right)}{4+y+z} = \frac{\exp\left((2-u)^2\right)}{4+u}.$$

Then we calculate the *weight function* $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right|$. First note that

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(u+v+2w) & \frac{1}{2}(u+v+2w) & u+v+2w \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} \\ &= (u+v+2w) \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}(u+v+2w). \end{aligned}$$

Since u, v, $w \geq 0$ in D, we see that the *weight function* is

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = +\frac{1}{2}\,(u+v+2w).$$

This is only 0 for $(u,v,w) = (0,0,0)$ in D, i.e. in just one point, which is a null-set (without a positive volume). Therefore we may continue with the transformation theorem in its abstract form:

$$\begin{aligned} I &= \int_A \frac{\exp\left((2-y-z)^3\right)}{4+y+z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_D \frac{\exp\left((2-u)^3\right)}{4+u}\,\frac{1}{2}\,(u+v+2w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \end{aligned}$$

Here it is obvious that we shall *not* start by integrating after u. If we choose u as the last (i.e. the outer) variable of integration, then we get by one of the reduction theorems that

$$(29.4)\quad I = \frac{1}{2}\int_0^2 \frac{\exp\left((2-u)^3\right)}{4+u}\left\{\int_{B(u)} (u+v+2w)\,\mathrm{d}v\,\mathrm{d}w\right\}\mathrm{d}u,$$

where $B(u)$ is the intersection of D for u constant, i.e.

$$B(u) = \left\{(v,w)\;\middle|\; 0 \leq v \leq 2-u,\, 0 \leq w \leq 1-\frac{u+v}{2}\right\}.$$

We calculate for fixed $u \in [0,2]$ the inner integral in (29.4) by first integrating vertically with respect to w:

$$\int_{B(u)} (u+v+2w)\,\mathrm{d}v\,\mathrm{d}w = \int_0^{2-u}\left\{\int_0^{1-\frac{u+v}{2}} (u+v+2w)\,\mathrm{d}w\right\}\mathrm{d}v.$$

We calculate the inner integral

$$\begin{aligned} \int_0^{1-\frac{u+v}{2}} (u+v+2w)\,\mathrm{d}w &= \left[(u+v)w+w^2\right]_0^{1-\frac{u+v}{2}} = \left[w(u+v+w)\right]_0^{1-\frac{u+v}{2}} \\ &= \left\{1-\frac{u+v}{2}\right\}\left\{1+\frac{u+v}{2}\right\} = 1-\frac{1}{4}\,(u+v)^2. \end{aligned}$$

By insertion we next get for fixed u that

$$\begin{aligned} \int_{B(u)} (u+v+2w)\,\mathrm{d}w\,\mathrm{d}v &= \int_0^{2-u}\left\{1-\frac{1}{4}\,(u+v)^2\right\}dv = \left[v-\frac{1}{12}\,(u+v)^3\right]_{v=0}^{2-u} \\ &= -(u-2)+\frac{1}{12}\,\left(u^3-2^3\right) \\ &= \frac{1}{12}\,(u-2)\left\{-12+u^2+2u+4\right\} \\ &= \frac{1}{12}\,(u-2)\left\{u^2+2u-8\right\} = \frac{1}{12}\,(u-2)^2(u+4). \end{aligned}$$

Note that we have found all factors. When this result is put into (29.4), we get the reduction

$$\begin{aligned} I &= \frac{1}{2}\int_0^2 \frac{\exp\left((2-u)^3\right)}{4+u}\left\{\int_{B(u)} (u+v+2w)\,\mathrm{d}v\,\mathrm{d}w\right\}\mathrm{d}u \\ &= \frac{1}{2}\int_0^2 \frac{\exp\left((2-u)^3\right)}{4+u}\cdot\frac{1}{12}\,(u-2)^2(u+4)\,\mathrm{d}u \\ &= \frac{1}{24}\int_0^2 \exp\left((2-u)^3\right)\,(u-2)^2\,\mathrm{d}u. \end{aligned}$$

Now, choose the substitution $t=(2-u)^3$. Then $\mathrm{d}t=-3(2-u)^2\,\mathrm{d}u$, and hence

$$(u-2)^2\,\mathrm{d}u = -\frac{1}{3}\,\mathrm{d}t.$$

Finally we get

$$\begin{aligned} I &= \frac{1}{24}\int_0^2 \exp\left((2-u)^3\right)\,(u-2)^2\,\mathrm{d}u = \frac{1}{24}\int_{(2-0)^3}^{(2-2)^3} \exp(t)\cdot\left(-\frac{1}{3}\right)\mathrm{d}t \\ &= \frac{1}{72}\int_0^8 e^t\,\mathrm{d}t = \frac{e^8-1}{72}. \end{aligned}$$

Remark 29.5 It is obvious from this example, that the application of transformation theorems is not an easy job. Therefore, one will usually be *given* the transform which should be applied,

$$u=f(x,y,z),\qquad v=g(x,y,z),\qquad w=h(x,y,z).$$

Then the task for the reader can be described in the following points:

1) *Solve* the equations after x, y, z, (from this follows automatically that the transform is *one-to-one*),

$$x=F(u,v,w),\qquad y=G(u,v,w),\qquad z=H(u,v,w).$$

2) *Identify* the parametric domain; use here the second main theorem and that a boundary in most cases by a continuous transform again is mapped into a part of the boundary.

3) *Calculate* the weight function $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ from the expressions found in 1).

4) *Reduce* the integrand

$$\Phi(x,y,z)=\Phi(F,G,H)=\Psi(u,v,w);$$

5) *Set up* the abstract reduction formula,

$$\int_A \Phi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_B \Psi(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

6) *Reduce* the right hand side of 5) by known methods, usually in rectangular coordinates, though semi-polar coordinates may occur, and calculate the value of the resulting integral.

It should be of no surprise that in general even this very simple type of example may be fairly large.$\Diamond$

Example 29.3 *Let B be the trapeze which is bounded by the coordinate axes and the lines given by the equations $x+y=1$ and $x+y=\frac{1}{2}$. Compute the plane integral*

$$\int_B \exp\left(\frac{y}{x+y}\right)\,\mathrm{d}x\,\mathrm{d}y$$

by introducing the new variable $(u,v)=(x+y,x-y)$.

A Transformation of a plane integral.

D Compute the Jacobian and find the new domain D.

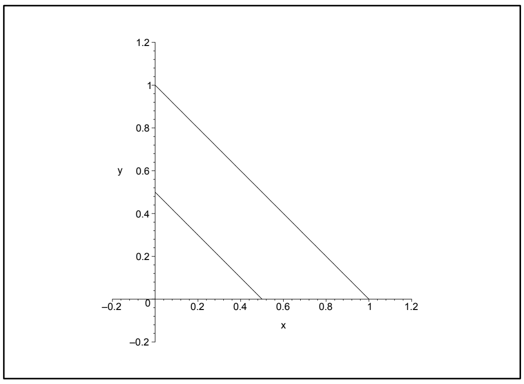

Figure 29.7: The domain B in the XY-plane.

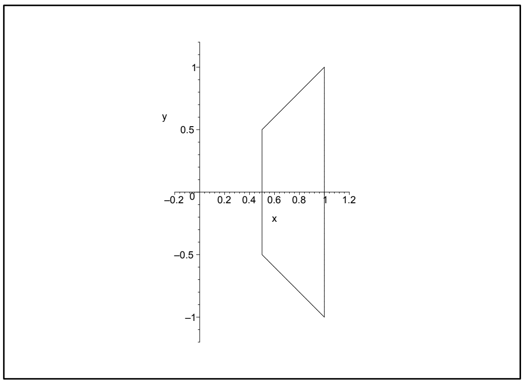

Figure 29.8: The domain D after the transformation to the UV-plane.

I From

$$(x,y)=\mathbf{\Phi}(u,v)=\left(\frac{u+v}{2},\frac{u-v}{2}\right),$$

follows that

$$J_{\mathbf{\Phi}} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & d\frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2},$$

and

$$D = \left\{ (u,v) \;\middle|\; \frac{1}{2} \le u \le 1, \; -u \le v \le u \right\}.$$

Then by the formula of transformation,

$$\begin{aligned} \int_B \exp\left(\frac{y}{x+y}\right) \mathrm{d}x\,\mathrm{d}y &= \int_D \exp\left(\frac{u-v}{2u}\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{2} \int_{\frac{1}{2}}^1 \sqrt{e} \left\{ \int_{-u}^u \exp\left(-\frac{v}{2u}\right) \mathrm{d}v \right\} \mathrm{d}u = \frac{\sqrt{e}}{2} \int_{\frac{1}{2}}^1 (-2u) \left[\exp\left(-\frac{v}{2u}\right) \right]_{v=-u}^u \mathrm{d}u \\ &= -\sqrt{e} \int_{\frac{1}{2}}^1 u \cdot \left(\frac{1}{\sqrt{e}} - \sqrt{e} \right) \mathrm{d}u = (e-1) \int_{\frac{1}{2}}^1 u\,\mathrm{d}u = \frac{3}{8}(e-1). \end{aligned}$$

Example 29.4 *Let B denote set in the first quadrant, which is bounded by the curves $xy = 1$ and $xy = 2$ and by the lines $y = x$ and $y = 4x$. Sketch B and compute the plane integral*

$$\int_B x^2y^2\,\mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables $(u,v) = \left(xy, \dfrac{y}{x}\right)$.

A Transformation of a plane integral.

D Sketch B. Find den inverse function

$$(x,y) = (x(u,v), y(u,v)) = \mathbf{\Phi}(u,v),$$

and find the corresponding domain D in the UV-plane. Calculate the Jacobian and finally transform the plane integral.

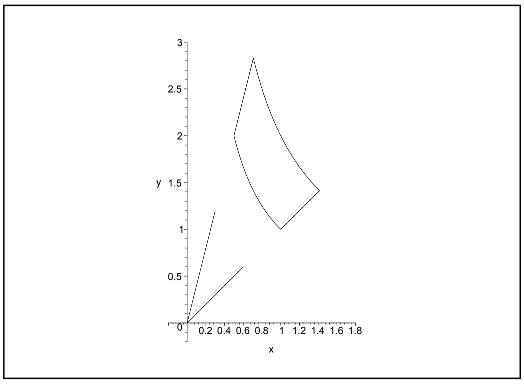

Figure 29.9: The domain B in the XY-plane.

I If $u = xy$ and $v = \dfrac{y}{x}$ and x, $y > 0$, then u, $v > 0$, and

$$x(u,v) = \sqrt{\frac{u}{v}}, \qquad y(u,v) = \sqrt{uv}.$$

The domain D is given by

$$1 \le xy = u \le 2 \quad\text{and}\quad 1 \le \frac{y}{x} = v \le 4,$$

hence

$$D = \{(u,v) \mid 1 \le u \le 2,\, 1 \le v \le 4\} = [1,2]\times[1,4],$$

i.e. a rectangle in the UV-plane, which it is no need to sketch.

Finally, the Jacobian is

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & -\dfrac{1}{2}\cdot\dfrac{1}{v}\sqrt{\dfrac{u}{v}} \\ \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\cdot\dfrac{1}{u}\sqrt{\dfrac{u}{v}} \end{vmatrix} = \frac{1}{4}\left\{\frac{1}{\sqrt{uv}}\sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}}\cdot\frac{1}{v}\sqrt{\frac{u}{v}}\right\} = \frac{1}{4}\left\{\frac{1}{v}+\frac{1}{v}\right\} = \frac{1}{2v}.$$

We get by the transformation formula of the plane integral

$$\begin{aligned}\int_B x^2y^2\,dx\,dy &= \int_D u^2\cdot\frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^2 u^2\,du\cdot\int_1^4\frac{1}{v}\,dv\\ &= \frac{1}{2}\left[\frac{1}{3}u^3\right]_1^2\cdot[\ln v]_1^4 = \frac{1}{6}(8-1)\ln 4 = \frac{7}{3}\ln 2.\end{aligned}$$

Example 29.5 *Find the area of the set in the first quadrant, which is bounded by the curves*

$$xy = 4,\qquad xy = 8,\qquad xy^3 = 5,\qquad xy^3 = 15,$$

by introducing the new variables $u = xy$ *and* $v = xy^3$.

A Area of a set computed by a transformation of a plane integral.

D Find the transformed domain D in the UV-plane and the inverse functions $x(u,v)$ and $y(u,v)$ by this transformation. Calculate the Jacobian and apply the transformation formula to find the area.

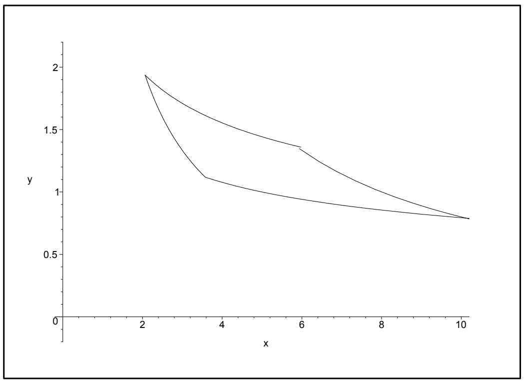

Figure 29.10: The domain D in the XY-plane. (Different scales on the axes).

I Let B be the given set in the first quadrant. Then x, $y > 0$ for $(x,y) \in B$. It follows immediately that we by the transformation get the domain

$$D = [4,8]\times[5,15].$$

From $u = xy$, $v = xy^3$, $u > 0$ and $v > 0$ follows $y^2 = \dfrac{v}{u}$ and $x^2 = \dfrac{u^3}{v}$, i.e.

$$y = +\sqrt{\frac{v}{u}},\qquad\text{and}\qquad x = +\sqrt{\frac{u^3}{v}}.$$

Then we get the Jacobian,

$$J(u,v) = \begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}\dfrac{3}{2}\sqrt{\dfrac{u}{v}} & -\dfrac{1}{2}\sqrt{\dfrac{u^3}{v^3}}\\[2mm] -\dfrac{1}{2}\sqrt{\dfrac{v}{u^3}} & \dfrac{1}{2}\sqrt{\dfrac{1}{uv}}\end{vmatrix} = \frac{3}{4}\,\frac{1}{v} - \frac{1}{4}\,\frac{1}{v} = \frac{1}{2v} > 0.$$

Hence the area is

$$\text{area}(B) = \int_B \mathrm{d}x\,\mathrm{d}y = \int_D J(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_4^8 \mathrm{d}u \cdot \int_4^1 5\frac{1}{2v}\,\mathrm{d}v = \frac{4}{2}[\ln v]_5^{15} = 2\ln 3.$$

Example 29.6 *Find the area of the set in the first quadrant, which is bounded by the curves*

$$y = x^3, \qquad y = 4x^3, \qquad x = y^3, \qquad x = 4y^3,$$

by introducing the new variables

$$u = \frac{y}{x^3}, \qquad v = \frac{x}{y^3}.$$

A Area of a set by a transformation of a plane integral.

D Sketch the domain B. Then find D and $x(u,v)$ and $y(u,v)$ by the transformation. Calculate the Jacobian and apply the transformation formula to find the area.

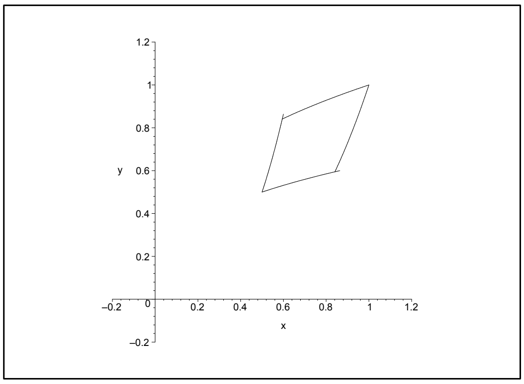

Figure 29.11: The domain B in the XY-plane.

I The curves $y = x^3$ and $x = y^3$ intersect at $(x,y) = (1,1)$. The curves $y = 4x^3$ and $x = 4y^3$ intersect at $(x,y) = \left(\frac{1}{2},\frac{1}{2}\right)$. It follows that if the transformation exists and is bijective, then

$$D = [1,4] \times [1,4].$$

Clearly, $x > 0$ and $y > 0$, and hence $u > 0$ and $v > 0$. We shall now try to solve the equations

$$u = \frac{y}{x^3} \quad \text{and} \quad v = \frac{x}{y^3} \qquad \text{for } u,\, v \in [1,4].$$

From

$$u^3 v = \frac{y^3}{x^9} \cdot \frac{x}{y^3} = \frac{1}{x^8}$$

follows that

$$x = u^{-\frac{3}{8}}\, v^{-\frac{1}{8}}, \qquad \text{and similarly } y = u^{-\frac{1}{8}}\, v^{-\frac{3}{8}}.$$

The Jacobian is

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2mm] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{3}{8}\, u^{-\frac{11}{8}}\, v^{-\frac{1}{8}} & -\dfrac{1}{8}\, u^{-\frac{9}{8}}\, v^{-\frac{3}{8}} \\[2mm] -\dfrac{1}{8}\, u^{-\frac{3}{8}}\, v^{-\frac{9}{8}} & -\dfrac{3}{8}\, u^{-\frac{1}{8}}\, v^{-\frac{11}{8}} \end{vmatrix} \\ &= \frac{9}{64}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}} - \frac{1}{64}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}} = \frac{1}{8}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}}. \end{aligned}$$

We get the area by applying the transformation formula

$$\begin{aligned} \operatorname{area}(B) &= \int_B \mathrm{d}S = \frac{1}{8} \int_1^4 u^{-\frac{3}{2}}\, \mathrm{d}u \cdot \int_1^4 v^{-\frac{3}{2}}\, \mathrm{d}v = \frac{1}{8} \left\{ \int_1^4 t^{-\frac{3}{2}}\, \mathrm{d}t \right\}^2 \\ &= \frac{1}{8} \left\{ \left[-\frac{2}{\sqrt{t}} \right]_1^4 \right\}^2 = \frac{1}{8}\,(2-1)^2 = \frac{1}{8}. \end{aligned}$$

Example 29.7 *Let $B \subset \mathbb{R}^2$ be given by*

$$0 \le x, \qquad 0 \le y, \qquad \sqrt{x} + \sqrt{y} \le 1.$$

find the area of B and the plane integral

$$I = \int_B \exp\left[\left(\sqrt{x} + \sqrt{y}\right)^4\right] \mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables

$$u = \sqrt{x} + \sqrt{y}, \qquad v = \sqrt{x} - \sqrt{y}.$$

A Transformation of a plane integral.

D Sketch B; find x and y as functions of u and v; calculate the Jacobian; find the domain of the parameters $(u, v) \in A$; finally, apply the transformation theorem.

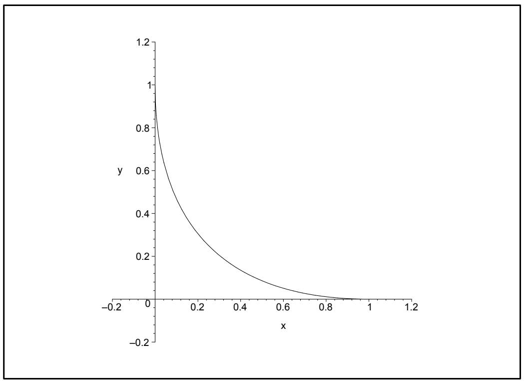

Figure 29.12: The domain A in the (X, Y)-plane.

I If we put $u = \sqrt{x} + \sqrt{y}$ and $v = \sqrt{x} - \sqrt{y}$, then

$$2\sqrt{x} = u + v \qquad \text{and} \qquad 2\sqrt{y} = u - v,$$

hence

$$x = \frac{1}{4}(u+v)^2 \qquad \text{and} \qquad y = \frac{1}{2}(u-v)^2.$$

Then we get the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(u+v) & \frac{1}{2}(u+v) \\ \frac{1}{2}(u-v) & -\frac{1}{2}(u-v) \end{vmatrix}$$

$$= \frac{1}{2}(u+v) \cdot \frac{1}{2}(u-v) \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -\frac{1}{2}\left(u^2 - v^2\right).$$

We shall then find the domain of the new parameters A:

1) The boundary part $x = 0$ corresponds to $u + v = 0$.
2) The boundary part $y = 0$ corresponds to $u - v = 0$.
3) The boundary part $\sqrt{x} + \sqrt{y} = 1$ corresponds to $u = 1$.

Since a closed and bounded set by the second main theorem of continuous functions is mapped into a closed and bounded set by this continuous change of variables, the new domain is the triangle A on the figure.

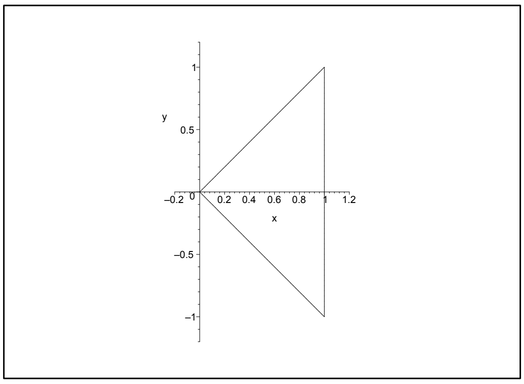

Figure 29.13: The domain A in the (U, V)-plane.

Note that the Jacobian is *negative* on A, so this time we shall need the absolute values in the formula.

By the transformation theorem,

$$\begin{aligned}
\text{area}(B) &= \int_B dx\, dy = \int_A \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \frac{1}{2} \int_A (u^2 - v^2)\, du\, dv \\
&= \frac{1}{2} \int_0^1 \left\{ \int_{-u}^{u} (u^2 - v^2)\, dv \right\} du = \frac{1}{2} \int_0^1 \left[u^2 v - \frac{1}{3} v^3 \right]_{-u}^{u} du \\
&= \frac{1}{2} \int_0^1 \left(2u^3 - \frac{2}{3} u^3 \right) du = \frac{2}{3} \int_0^1 u^3\, du = \frac{1}{6},
\end{aligned}$$

and

$$\begin{aligned}
I &= \int_B \exp\left[\left(\sqrt{x} + \sqrt{y} \right)^4 \right) dx\, dy = \frac{1}{2} \int_A \exp\left(u^4\right) \cdot \left(u^2 - v^2\right) du\, dv \\
&= \frac{1}{2} \int_0^1 \left\{ \int_{-u}^{u} \exp\left(u^4\right) \cdot \left(u^2 - v^2\right) dv \right\} du = \frac{2}{3} \int_0^1 \exp\left(u^4\right) \cdot u^3\, du \\
&= \frac{1}{6} \int_0^1 e^t\, dt = \frac{e-1}{6}.
\end{aligned}$$

Example 29.8 *Define a vector field* $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ *in the following way,*

$$\mathbf{r}(u, v) = (e^u \cos v, e^u \sin v) .$$

Prove that the Jacobian $J_{\mathbf{r}}$ *is different from zero almost everywhere, and that* $\mathbf{r}$ *is not injective.*

A Jacobian and a non-injective transformation.

D Calculate and exploit the periodicity.

I The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\[2mm] e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u} \neq 0.$$

Then note that $(u, v) = (u_0, v_0 + 2p\pi)$, $p \in \mathbb{Z}$, are all mapped into the same point

$$(x, y) = (e^{u_0} \cos v_0, e^{u_0} \sin v_0) ,$$

so the transformation is not injective.

REMARK. We may add that $\mathbb{R}^2$ by $\mathbf{r}$ is mapped (infinitely often) onto $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Example 29.9 *Define a vector field* $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ *as follows:*

$$\mathbf{r}(u, v) = \left(u^2 - v^2, 2uv\right) .$$

Prove that the Jacobian $J_{\mathbf{r}}$ *is different from zero almost everywhere ant that* $\mathbf{r}$ *is not injective.*

A Jacobian and a non-injective transformation.

D Calculate the Jacobian and find two different (u, v)-points which are mapped into the same (x, y).

I The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\[2mm] 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right) \neq 0 \qquad \text{for } (u, v) \neq (0, 0).$$

Clearly, (u, v) and $(-u, -v)$ are mapped into the same point,

$$(x, y) = \left(u^2 - v^2, 2uv\right) ,$$

so the map is not injective for $(u, v) \neq (0, 0)$.

Example 29.10 *Let B be the parallelogram of vertices $(0,0)$, $(1,-1)$, $(2,1)$ and $(3,0)$. Compute the plane integral*

$$I = \int_B \frac{\cos\left(\frac{1}{2}\,\pi(x+y)\right)}{1+x-2y}\,\mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables

$$u = x+y, \qquad v = x-2y.$$

A Plane integral by a change of variables and the transformation formula.

D Sketch B and find the domain D. Calculate the Jacobian and insert into the formula.

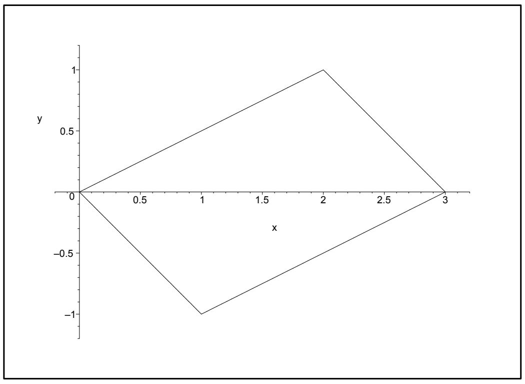

Figure 29.14: The parallelogram B.

I It follows from the figure that

$$u = x + y \in [0,3] \qquad \text{and} \qquad v = x - 2y \in [0,3],$$

and the new domain is the square $D = [0,3] \times [0,3]$.

From

$$x = \frac{2}{3}\,u + \frac{1}{3}\,v \qquad \text{and} \qquad y = \frac{1}{3}\,u - \frac{1}{3}\,v,$$

follows that the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9}.$$

When we finally put everything into the transformation formula, then

$$\begin{aligned} I &= \int_B \frac{\cos(\frac{1}{2}\,\pi(x+y))}{1+x-2y}\,\mathrm{d}x\,\mathrm{d}y = \int_D \frac{\cos(\frac{1}{2}\,\pi \cdot u)}{1+v}\left|-\frac{1}{9}\right|\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{9}\int_0^3 \cos\left(\frac{\pi}{2}\,u\right)\,\mathrm{d}u \cdot \int_0^3 \frac{\mathrm{d}v}{1+v} = \frac{1}{9}\cdot\frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\,u\right)\right]_0^3 \cdot [\ln(1+v)]_0^3 \\ &= \frac{2}{9\pi}\left\{\sin\left(\frac{3\pi}{2}\right) - 0\right\} \cdot \{\ln 4 - \ln 1\} = -\frac{4}{9\pi}\,\ln 2. \end{aligned}$$

Example 29.11 *Let B be the plane set which is bounded by the X-axis and the line of equation $y = x$ and an arc of the parabola given by*

$$5x = 4 + y^2, \qquad y \in [0, 1].$$

Calculate the plane integral

$$I = \int_B \cos\left[\left(\sqrt{\frac{5}{4}\,x + y} + \sqrt{\frac{5}{4}\,x - y}\right)^4\right] \mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables (u, v) given by

$$5x = u^2 + v^2, \qquad 2y = uv, \qquad -u \le v \le u.$$

A Plane integral by a change of variables and the transformation formula.

D Sketch B and find the new domain D. Calculate the Jacobian and put everything into the formula.

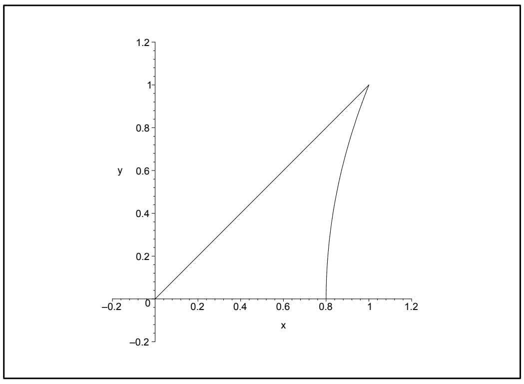

Figure 29.15: The point set B.

I It follows from $5x = u^2 + v^2$ and $2y = uv$ that

$$5x + 4y = u^2 + v^2 + 2uv = (u + v)^2, \qquad 5x - 4y = u^2 + v^2 - 2uv = (u - v)^2.$$

Since $|v| \le u$, we get from this

$$u + v = +\sqrt{5x + 4y} \quad \text{and} \quad u - v = +\sqrt{5x - 4y},$$

hence

$$u = \frac{\sqrt{5x + 4y} + \sqrt{5x - 4y}}{2} \quad \text{and} \quad v = \frac{\sqrt{5x + 4y} - \sqrt{5x - 4y}}{2}.$$

Then we determine the boundary curves of the new domain.

1) If $y = x$, $x \in [0,1]$, then

$$u = \frac{\sqrt{9x}+\sqrt{x}}{2} = 2\sqrt{x} \quad \text{and} \quad v = \frac{\sqrt{9x}-\sqrt{x}}{2} = \sqrt{x},$$

so this boundary curve is transformed into $v = \frac{1}{2}\,u$. Then by a small consideration, $u \in [0,2]$.

2) If $y = 0$, $x \in \left[0, \frac{4}{5}\right]$ on the X-axis, then $v = 0$ and $u = \sqrt{5x} \in [0,2]$.

3) If finally $5x = 4 + y^2$, $y \in [0,1]$, then

$$4 + y^2 = u^2 + v^2 \qquad \text{and} \qquad 4y = 2uv,$$

i.e.

$$(u+v)^2 = (y+2)^2 \qquad \text{and} \qquad (u-v)^2 = (2-y)^2,$$

thus

$$u + v = y + 2 \geq 0 \qquad \text{and} \qquad u - v = 2 - y \geq 0,$$

or $u = 2$ and $v = y \in [0,1]$. Then we can sketch the new domain (a triangle).

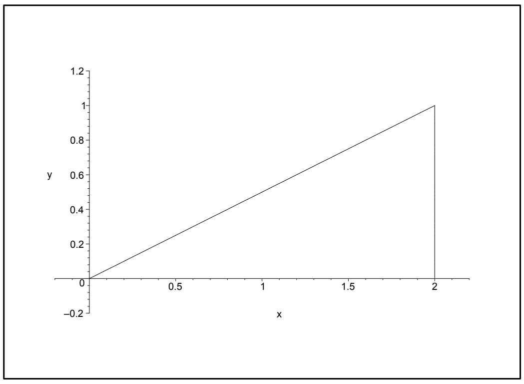

Figure 29.16: The new domain D.

Since

$$x = \frac{1}{5}\,(u^2+v^2), \qquad y = \frac{1}{2}\,uv,$$

we get the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5}\,u & \frac{2}{5}\,v \\ \frac{1}{2}\,v & \frac{1}{2}\,u \end{vmatrix} = \frac{1}{5}\,\left(u^2 - v^2\right) \geq 0.$$

Finally, since

$$\frac{5}{4}\,x + y = \frac{1}{4}\,(5x+4y) = \frac{1}{4}\,\left(u^2+v^2+2uv\right) = \left(\frac{u+v}{2}\right)^2,$$

and similarly,

$$\frac{5}{4}\,x-y=\left(\frac{u-v}{2}\right)^2,$$

we get the plane integral

$$\begin{aligned}
I &= \int_B \cos\left[\left(\sqrt{\frac{5}{4}\,x+y}+\sqrt{\frac{5}{4}\,x-y}\right)^4\right]\,dx\,dy\\
&= \int_D \cos\left[\left(\frac{u+v}{2}+\frac{u-v}{2}\right)^4\right]\cdot\frac{1}{5}\left(u^2-v^2\right)\,du\,dv = \int_D \cos\left(u^4\right)\cdot\frac{1}{5}\left(u^2-v^2\right)\,du\,dv\\
&= \int_0^2 \cos\left(u^4\right)\left\{\int_0^{\frac{u}{2}}\frac{1}{5}\left(u^2-v^2\right)\,dv\right\}du = \int_0^2 \cos\left(u^4\right)\cdot\frac{1}{5}\left\{u^2\cdot\frac{u}{2}-\frac{1}{3}\cdot\left(\frac{u^3}{8}\right)\right\}du\\
&= \frac{1}{5}\cdot\frac{11}{24}\int_0^2 \cos\left(u^4\right)\,u^3\,du = \frac{11}{480}\left[\sin\left(u^4\right)\right]_0^2 = \frac{11}{480}\,\sin 16.
\end{aligned}$$

Example 29.12 *Let B be the plane point set which is bounded by the X-axis and the line of equation $y = \frac{1}{2}\,x$, and the branches of the hyperbola,*

$$x^2 - y^2 = 1, \quad x > 0, \quad \text{and} \quad x^2 - y^2 = 4, \quad x > 0.$$

Compute the plane integral

$$I = \int_B \frac{x+y}{x-y} \exp\left(x^2 - y^2\right) \, dx \, dy$$

by introducing the new variables (u, v) given by

$$x = u \cosh v, \qquad y = u \sinh v.$$

A Transformation of a plane integral.

D Sketch the domain B, and find the domain D of the new variables, and compute the Jacobian. Finally, insert everything into the transformation formula.

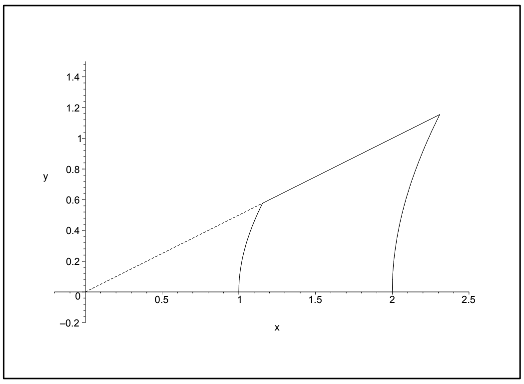

Figure 29.17: The domain B.

I If $y = 0$, $x > 0$, then $v = 0$ and $x = u$, hence the segment on the X-axis is transformed onto a segment on the U-axis.

If $y = \frac{1}{2}\,x$, then $u \sinh v = \frac{1}{2}\,u \cosh v$, i.e.

$$\tanh v = \frac{1}{2} = \frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{e^{2v} - 1}{e^{2v} + 1},$$

or $e^{2v} + 1 = 2\,e^{2v} - 2$, thus $e^{2v} = 3$, and hence $v = \frac{1}{2}\,\ln 3$, and u is a "free" variable.

If $x^2 - y^2 = 1$ $x > 0$, then $u^2 = 1$, and since $u > 0$, we must have $u = 1$.

If $x^2 - y^2 = 4$, $x > 0$, then $u^2 = 4$, and since $u > 0$, we must have $u = 2$.

Summarizing, the new domain is the rectangle

$$D = [1,2] \times \left[0, \frac{1}{2}\ln 3\right].$$

Then the Jacobian is computed,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \end{vmatrix} = u > 0.$$

By the transformation formula,

$$\begin{aligned} I &= \int_B \frac{x+y}{x-y} \exp\left(x^2-y^2\right) \mathrm{d}x\,\mathrm{d}y = \int_D \frac{u(\cosh v + \sinh v)}{u\cosh v - \sinh v} \exp\left(u^2\right) \cdot u \,\mathrm{d}u\,\mathrm{d}v \\ &= \int_D e^{2v} \exp\left(u^2\right) u \,\mathrm{d}u\,\mathrm{d}v = \int_1^2 \exp\left(u^2\right) u \,\mathrm{d}u \cdot \int_0^{\frac{1}{2}\ln 3} e^{2v}\,\mathrm{d}v \\ &= \frac{1}{2}\left[\exp\left(u^2\right)\right]_1^2 \cdot \frac{1}{2}\left[e^{2v}\right]_0^{\frac{1}{2}\ln 3} = \frac{1}{4}\left(e^4-e\right)\cdot(3-1) = \frac{e}{2}\left(e^3-1\right). \end{aligned}$$

Example 29.13 *A triangle B in the (X,Y)-plane is given by the inequalities*

$$x+y \geq 1, \qquad 2y - x \leq 2, \qquad y - 2x \geq -2.$$

By introducing

(29.5) $\quad u = x+y, \qquad v = x-y,$

we get a map from the (X,Y)-plane onto the (U,V)-plane.

1) Prove that the image D in the (U,V)-plane of B by this map is given by

$$1 \leq u \leq 4, \qquad u-4 \leq 3v \leq 4-u,$$

and sketch D.

2) Calculate the plane integral

$$\int_B \frac{3}{x+y}\,\mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables given by (29.5).

A Transformation of a plane integral.

D Find the domain of the new variables D and compute the Jacobian, and then finally insert into the formula.

I 1) It follows from (29.13) that

$$x = \frac{u+v}{2} \qquad \text{and} \qquad y = \frac{u-v}{2}.$$

Hence

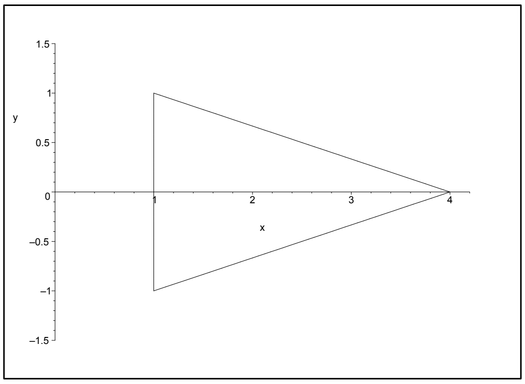

Figure 29.18: The new domain D.

a) $x+y \geq 1$ is transformed into $u \geq 1$,
b) $2y-x \leq 2$ is transformed into $u-3v \leq 4$,
c) $y-2x \geq -2$ is transformed into $u+3v \leq 4$.

We get by a rearrangement, $u-4 \leq 3v \leq 4-u$, hence $u \leq 4$, and

$$D = \{(u,v) \mid 1 \leq u \leq 4,\; u-4 \leq 3v \leq 4-u\}.$$

We can here exploit that it is given that B is a triangle and thus bounded. The transformation (29.13) is continuous, so D is connected an bounded, and then we can sketch the three boundary lines and identify the image as the bounded part.

2) The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Then by the transformation formula,

$$\begin{aligned} \int_B \frac{3}{x+y}\,dx\,dy &= \int_D \frac{3}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{3}{2}\int_1^4 \left\{\int_{-\frac{4-u}{3}}^{\frac{4-u}{3}} dv\right\} du \\ &= \frac{3}{2}\int_1^4 \frac{1}{u}\cdot 2\cdot\frac{4-u}{3}\,du = \int_1^4 \left(\frac{4}{u}-1\right) du = 4\ln 4 - 3 = 8\ln 2 - 3. \end{aligned}$$

Example 29.14 *Let B be the bounded domain which is given by the inequalities*

$$e^{-x} \le y \le 2e^{-x}, \qquad e^{x} \le y \le^{2} e^{x}.$$

1. *Sketch B.*

If we put

$$u = y\,e^{x}, \qquad v = y\,e^{-x}, \tag{29.6}$$

we get a map of the (X,Y)-plane into the (U,V)-plane.

2. *Prove that the image of B by this map is the square $[1,2]\times[1,2]$.*

3. *Calculate the plane integral*

$$I = \int_B 4y^2 \exp\left(y^2 + x\right)\,\mathrm{d}x\,\mathrm{d}y$$

by introducing the new variables given by (29.6).

A Transformation of a plane integral.

D Follow the guidelines supplied by a calculation of the Jacobian before everything is put into the transformation formula.

ALTERNATIVELY, one can actually in this case compute the plane integral directly.

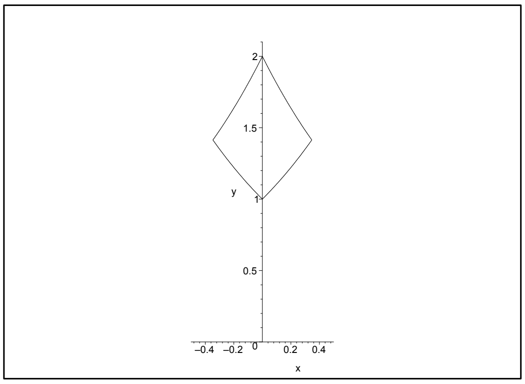

Figure 29.19: The domain B.

I 1) Let us first find the intersection point of the boundary curves of B.

a) If $y = e^{-x} = 2e^{x}$, then $x = -\dfrac{1}{2}\ln 2$ and hence $y = \sqrt{2}$.

b) If $y = e^{x} = 2e^{-x}$, then $x = \dfrac{1}{2}\ln 2$ and hence $y = \sqrt{2}$.

c) The remaining two intersection points are immediately seen to be $(0,1)$ and $(0,2)$.

Then it is easy to sketch the domain B, even if one does not have MAPLE at hand.

2) By the change of variables $u = y\,e^{x}$ and $v = y\,e^{-x}$,

a) $y = e^{x}$ and $x \in \left[0, \dfrac{1}{2}\ln 2\right]$ is transformed into $v = 1$ and $u = e^{2x} \in [1,2]$,

b) $y = 2e^{-x}$ and $x \in \left[0, \dfrac{1}{2}\ln 2\right]$ is transformed into $u = 2$ and $v = 2e^{2x} \in [1,2]$,

c) $y = 2e^{x}$ and $x \in \left[-\dfrac{1}{2}\ln 2, 0\right]$ is transformed into $v = 2$ and $u = 2e^{2x} \in [1,2]$,

d) $y = e^{-x}$ and $x \in \left[-\dfrac{1}{2}\ln 2, 0\right]$ is transformed into $u = 1$ and $v = e^{-2x} \in [1,2]$.

Thus we get the new domain $D = [1,2]\times[1,2]$ in the (U,V)-plane.

3) Then we find x and y as functions of u and v:

From $y \geq 1$ and $u,\, v \geq 1$, follows that

$$\frac{u}{v} = e^{2x}, \quad \text{dvs.} \quad x = \frac{1}{2}\ln\left(\frac{u}{v}\right) = \frac{1}{2}\ln u - \frac{1}{2}\ln v.$$

From $v = y\,e^{-x}$, follows that

$$y = v\,e^{x} = v\sqrt{\frac{u}{v}} = \sqrt{uv}.$$

This gives the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2u} & -\frac{1}{2v} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{4}\left(\frac{1}{\sqrt{uv}} + \frac{1}{\sqrt{uv}}\right) = \frac{1}{2}\,\frac{1}{\sqrt{uv}} > 0.$$

When we insert into the transformation formula, we get

$$\begin{aligned}
I &= \int_B 4y^2\exp\left(y^2+x\right)\,dx\,dy = \int_D 4\,uv\,\exp\left(uv+\frac{1}{2}\,\ln\left(\frac{u}{v}\right)\right)\frac{1}{2}\,\frac{1}{\sqrt{uv}}\,du\,dv \\
&= \int_D 4uv\,\exp(uv)\cdot\sqrt{\frac{u}{v}}\cdot\frac{1}{2}\,\frac{1}{\sqrt{uv}}\,du\,dv = \int_D 2u\,\exp(uv)\,du\,dv \\
&= 2\int_1^2\left\{\int_1^2 u\,\exp(uv)\,dv\right\}\,du = 2\int_1^2 [\exp(uv)]_{v=1}^2\,du \\
&= 2\int_1^2\left(e^{2u}-e^u\right)\,du = \left[e^{2u}-2e^u\right]_1^2 = e^4-2e^2-e^2+2e = e^4-3e^2+2e.
\end{aligned}$$

Alternatively, it is actually possible to calculate the plane integral directly without using the transformation theorem. First write $B = B_1 \cup B_2$, as an (almost) disjoint union where

$$B_1 = \left\{(x,y)\;\middle|\;\sqrt{2}\le y\le 2,\,\ln\left(\frac{y}{2}\right)\le x\le\ln\left(\frac{2}{y}\right)\right\}$$

and

$$B_2 = \left\{(x,y)\;\middle|\;1\le y\le\sqrt{2},\,\ln\left(\frac{1}{y}\right)\le x\le\ln y\right\}.$$

We have the following natural splitting,

$$\int_B 4y^2\,\exp\left(y^2+x\right)\,dx\,dy = I_1+I_2,$$

where

$$\begin{aligned}
I_1 &= \int_{B_1} 4y^2\,\exp\left(y^2+x\right)\,dx\,dy = \int_{\sqrt{2}}^2\left\{\int_{\ln\left(\frac{y}{2}\right)}^{\ln\left(\frac{2}{y}\right)} 4y^2\,\exp\left(y^2\right)e^x\,dx\right\}\,dy \\
&= \int_{\sqrt{2}}^2 4y^2\,\exp\left(y^2\right)\cdot\left(\frac{2}{y}-\frac{y}{2}\right)\,dy = \int_{\sqrt{2}}^2\left(8y-2y^3\right)\,\exp\left(y^2\right)\,dy \\
&= \int_{\sqrt{2}}^2 2\left(4-y^2\right)\,\exp\left(y^2\right)\,2y\,dy = \int_{t=2}^4 (4-t)e^t\,dt = \left[(5-t)e^t\right]_2^4 = e^4-3e^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= int_{B_2} 4y^2\,\exp\left(y^2+x\right)\,dx\,dy = \int_1^{\sqrt{2}}\left\{\int_{\ln\left(\frac{1}{y}\right)}^{\ln y} 4y^2\,\exp\left(y^2\right)e^x\,dx\right\}\,dy \\
&= \int_1^{\sqrt{2}} 4y^2\,\exp\left(y^2\right)\cdot\left(y-\frac{1}{y}\right)\,dy = \int_1^{\sqrt{2}}\left(4y^3-4y\right)\,\exp\left(y^2\right)\,dy \\
&= \int_1^{\sqrt{2}}\left(2y^2-2\right)\,\exp\left(y^2\right)\cdot 2y\,dy = 2\int_1^2 (t-1)e^t\,dt = 2\left[(t-2)e^t\right]_1^2 = 2e.
\end{aligned}$$

Summarizing we get

$$\int_B 4y^2\,\exp\left(y^2+x\right)\,dx\,dy = I_1+I_2 = e^4-3e^2+2e.$$

Example 29.15 *Let A denote the tetrahedron, which is bounded by the four planes of the equations*

$$x+y=1, \quad y+z=1, \quad z+x=1, \quad x+y+z=1.$$

Calculate the space integral

$$I=\int_A (x+y)(y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

by introducing the new variables

$$u=1-x-y, \quad v=1-y-z, \quad w=1-z-x.$$

A Transformation of a space integral.

Find the Jacobian and the limits of u, v and w.

I We derive from

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w=\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

and

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\left|\begin{array}{rrr} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{array}\right| = -1-1+0-0-0-0=-2$$

that the weight function is

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}=\frac{1}{2}.$$

The integrand is

$$(x+y)(y+z)=(1-u)(1-v).$$

Considering the limits of u, v and w we see that

$$\begin{array}{lll} x+y=1 & \text{corresponds to} & u=1-x-y=0, \\ y+z=1 & \text{corresponds to} & v=1-y-z=0, \\ z+x=1 & \text{corresponds to} & w=1-z-x=0. \end{array}$$

From

$$u+v+w=3-2(x+y+z),$$

follows that

$$x+y+z=1 \quad \text{corresponds to} \quad u+v+w=1.$$

Finally, the tetrahedron lies in the first octant of the XYZ-space, where $x+y\le 1$, $y+z\le 1$ and $z+x\le 1$. Hence the domain in the UVW-space is

$$\begin{aligned} B &= \{(u,v,w)\mid u\ge 0,\, v\ge 0,\, w\ge 0,\, u+v+w\le 1\} \\ &= \{(u,v,w)\mid 0\le u\le 1,\, 0\le v\le 1-u,\, 0\le w\le 1-u-v\}. \end{aligned}$$

By this transformation followed by a reduction in rectangular coordinates we get

$$
\begin{aligned}
I &= \int_A (x+y)(y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_B (1-u)(1-v)\cdot\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}(1-v)\left\{\int_0^{1-u-v}\mathrm{d}w\right\}\mathrm{d}v\right\}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}(1-v)(1-u-v)\,\mathrm{d}v\right\}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}\left\{(v-1)^2+u(v-1)\right\}\mathrm{d}v\right\}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 (1-u)\left[\frac{1}{3}(v-1)^3+\frac{u}{2}(v-1)^2\right]_0^{1-u}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\frac{1}{3}-\frac{1}{3}u^23+\frac{u}{2}\cdot u^2-\frac{u}{2}\right\}\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\frac{1}{3}+\frac{1}{6}u^3-\frac{u}{2}\right\}du = \frac{1}{12}\int_0^1 (1-u)(2+u^3-3u)\,\mathrm{d}u \\
&= \frac{1}{12}\int_0^1 \left\{2+u^3-3u-2u-u^4+3u^2\right\}\mathrm{d}u = \frac{1}{12}\left\{2+\frac{1}{4}-\frac{3}{2}-\frac{2}{2}-\frac{1}{5}+\frac{3}{3}\right\} \\
&= \frac{1}{12}\cdot\frac{1}{30}(60+15-45-6) = \frac{1}{12}\cdot\frac{1}{30}\cdot 24 = \frac{1}{15}.
\end{aligned}
$$

Example 29.16 *Let A be the closed point set in $\mathbb{R}^3$, which is bounded by the four elliptic paraboloids of the equations*

$$(1) \quad z = \frac{3}{2} - \frac{1}{6}\,x^2 - \frac{2}{3}\,y^2$$

$$(2) \quad z = \frac{1}{2} - \frac{1}{2}\,x^2 - 2y^2,$$

$$(3) \quad z = -1 + \frac{1}{4}\,x^2 + y^2,$$

$$(4) \quad z = -2 + \frac{1}{8}\,x^2 + \frac{1}{2}\,y^2.$$

The point set A intersects the ZX-plane in a point set B_1, and the YZ-plane in a point set B_2.

1) Sketch B_1 and B_2.

2) Calculate the volume $Vol(A)$ and the space integral

$$I = \int_A \frac{1}{\sqrt{x^2+4y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

by introducing the new variables (u,v,w), such that

$$x = \sqrt{uv}\,\cos w, \quad y = \frac{1}{2}\,\sqrt{uv}\,\sin w, \quad z = \frac{1}{2}\,(u-v),$$

where

$$u,\, v \in [0,+\infty[, \qquad w \in [0,2\pi].$$

A Transformation of a space integral.

D First sketch B_1 (put $y=0$) and B_2 (put $x=0$). Then apply the transformation formula, i.e. calculate the weight function and change variables.

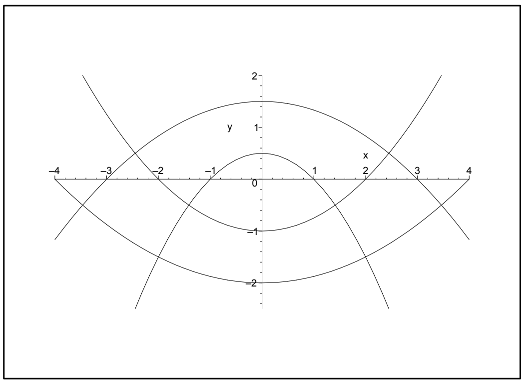

Figure 29.20: The set B_1 is the union of the two "oblique" quadrilateral sets.

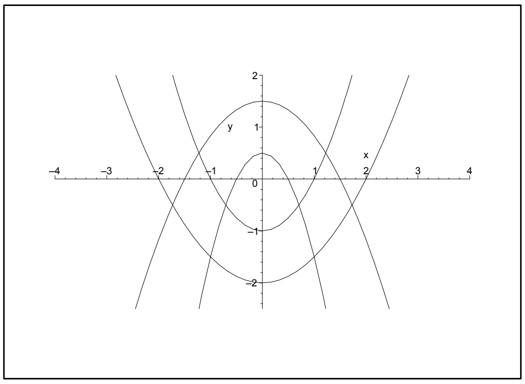

Figure 29.21: The set B_2 is the union of the two "oblique" quadrilateral sets.

I 1) By putting $y=0$, we get in the XZ-plane the four parabolas

$$z=\frac{3}{2}-\frac{1}{6}\,x^2,\quad z=\frac{1}{2}-\frac{1}{2}\,x^2,\quad z=-1+\frac{1}{4}\,x^2,\quad z=-2+\frac{1}{8}\,x^2,$$

and it is easy to sketch B_1.

By putting $x=0$, we get in the YZ-plane

$$z=\frac{3}{2}-\frac{2}{3}\,y^2,\quad z=\frac{1}{2}-2y^2,\quad z=-1+y^2,\quad z=-2+\frac{1}{2}\,y^2,$$

and it is easy to sketch B_2.

2) Let

$$x=\sqrt{uv}\,\cos w,\quad y=\frac{1}{2}\,\sqrt{uv}\,\sin w,\quad z=\frac{1}{2}\,(u-v),$$

where u, $v\ge 0$ and $w\in[0,2\pi]$. We shall first find the image of A by this transformation.

a) By insertion into the boundary surface

$$z=\frac{3}{2}-\frac{1}{6}\,x^2-\frac{2}{3}\,y^2$$

we get

$$\frac{1}{2}\,(u-v)=\frac{3}{2}-\frac{1}{6}\,uv\cos^2 w-\frac{1}{6}\,uv\sin^2 w=\frac{3}{2}-\frac{1}{6}\,uv,$$

hence $3(u-v)=9-uv$, which is reformulated as

$$uv+3u=u(v+3)=9+3v=3(v+3).$$

It follows from $v\ge 0$ that $u=3$, hence this boundary surface is mapped into a part of the plane $u=3$.

b) By insertion into the boundary surface

$$z = \frac{1}{2} - \frac{1}{2}\,x^2 - 2y^2$$

we get

$$\frac{1}{2}\,(u-v) = \frac{1}{2} - \frac{1}{2}\,uv\cos^2 w - \frac{1}{2}\,uv\sin^2 w = \frac{1}{2} - \frac{1}{2}\,uv,$$

i.e. $u - v = 1 - uv$, and thus

$$uv + u = u(v+1) = v+1.$$

From $v \geq 0$ follows that $u = 1$, hence the boundary surface is mapped into a part of the plane $u = 1$.

c) We get by insertion into the boundary surface

$$z = -1 + \frac{1}{4}\,x^2 + y^2$$

that

$$\frac{1}{2}\,(u-v) = -1 + \frac{1}{4}\,uv\cos^2 w + \frac{1}{4}\,uv\sin^2 w = -1 + \frac{1}{4}\,uv,$$

thus $2(u-2) = uv - 4$, and hence

$$uv + 2v = v(u+2) = 2u + 4 = 2(u+2).$$

It follows from $u \geq 0$ that $v = 2$, so the boundary surface is mapped into a part of the plane $v = 2$.

d) We get by insertion into the boundary surface

$$z = -2 + \frac{1}{8}\,x^2 + \frac{1}{2}\,y^2$$

that

$$\frac{1}{2}\,(u-v) = -2 + \frac{1}{8}\,uv\cos^2 w + \frac{1}{8}\,uv\sin^2 w = -2 + \frac{1}{8}\,uv,$$

thus $4(u-v) = -16 + uv$, and hence

$$uv + 4v = v(u+4) = 4u + 16 = 4(u+4).$$

It follows from $u \geq 0$ that $v = 4$, so the boundary surface is mapped into a part of the plane $v = 4$.

The only condition on w is that $(\cos w, \sin w)$ shall encircle the unit circle only once, so $w \in [0, 2\pi]$.

By the transformation A is mapped onto the set

$$B = [1,3] \times [2,4] \times [0,2\pi].$$

Then we calculate the Jacobian (for u, $v > 0$)

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{1}{2}\sqrt{\frac{v}{u}}\,\cos w & \frac{1}{2}\sqrt{\frac{u}{v}}\,\cos w & -\sqrt{uv}\,\sin w \\ \frac{1}{4}\sqrt{\frac{v}{u}}\,\sin w & \frac{1}{4}\sqrt{\frac{u}{v}}\,\sin w & \frac{1}{2}\sqrt{uv}\,\cos w \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} \\
&= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\sqrt{uv}\begin{vmatrix} \sqrt{\frac{v}{u}}\,\cos w & \sqrt{\frac{u}{v}}\,\cos w & -2\sin w \\ \sqrt{\frac{v}{u}}\,\sin w & \sqrt{\frac{u}{v}}\,\sin w & 2\cos w \\ 1 & -1 & 0 \end{vmatrix} \\
&= \frac{1}{16}\sqrt{uv}\cdot 2\begin{vmatrix} \sqrt{\frac{v}{u}}\,\cos w & \left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\cos w & -\sin w \\ \sqrt{\frac{v}{u}}\,\sin w & \left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\sin w & \cos w \\ 1 & 0 & 0 \end{vmatrix} \\
&= \frac{1}{8}\sqrt{uv}\left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\begin{vmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{vmatrix} = \frac{1}{8}(u+v) > 0.
\end{aligned}$$

We get by the transformation formula,

$$\begin{aligned}\mathrm{Vol}(A) &= \int_A \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_B \frac{\partial(x,y,z)}{\partial(u,v,w)}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \int_0^{2\pi}\left\{\int_2^4\left\{\int_1^3 \frac{1}{8}\,(u+v)\,\mathrm{d}u\right\}\mathrm{d}v\right\}\mathrm{d}w \\ &= \frac{1}{8}\cdot 2\pi\int_2^4\left[\frac{u^2}{2}+uv\right]_{u=1}^3 dv = \frac{\pi}{8}\int_2^4\{9+6v-1-2v\}\,\mathrm{d}v \\ &= \frac{\pi}{8}\int_2^4\{4v+8\}\,\mathrm{d}v = \frac{\pi}{8}\left[2v^2+8v\right]_2^4 = \frac{\pi}{4}\left[v^2+4v\right]_2^4 \\ &= \frac{\pi}{4}\{16+16-4-8\} = \pi\{4+4-1-2\} = 5\pi.\end{aligned}$$

Let us turn to the space integral. Since

$$x^2+4y^2+z^2 = uv\cos^2 w + uv\sin^2 w + \frac{1}{4}\,(u-v)^2 = \frac{1}{4}\left\{(u-v)^2+4uv\right\} = \frac{1}{4}\,(u+v)^2,$$

and $u+v>0$, the integrand is transformed into

$$\frac{1}{\sqrt{x^2+4y^2+z^2}} = \frac{2}{u+v}.$$

Finally, by the transformation formula,

$$\begin{aligned}&\int_A \frac{1}{\sqrt{x^2+4y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_B \frac{1}{u+v}\cdot\frac{\partial(x,y,z)}{\partial(u,v,w)}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &\quad= \int_B \frac{2}{u+v}\cdot\frac{u+v}{8}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \frac{1}{4}\int_B \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \frac{1}{4}\,\mathrm{Vol}(B) \\ &\quad= \frac{1}{4}\cdot 2\cdot 2\cdot 2\pi = 2\pi.\end{aligned}$$

Example 29.17 *We can write the formula of transformation of a space integral in the following way,*

$$\int_{\tilde{\Omega}} f(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\Omega} = \int_{\Omega} f(\tilde{\mathbf{x}}(\mathbf{x}))\,|J(\mathbf{x})|\,\mathrm{d}\Omega,$$

where J is the determinant of that matrix, the elements of which are

$$\frac{\partial \tilde{\mathbf{x}}_i}{\partial x_j}, \qquad i,\, j \in \{1,2,3\}.$$

We shall in particular interpret $\tilde{\Omega}$ as created from Ω by a translation *and a* deformation. *This means that to the point $\mathbf{x}$ we let correspond a point $\tilde{\mathbf{x}}$ given by*

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{u}(\mathbf{x}),$$

where $\mathbf{u}$ is the displacement vector field. *When $\mathbf{u}$ is a constant vector, we get a translation. However, in general $\mathbf{u}$ varies in space (as indicated by the notation), such that we have a combination of a translation and a deformation.*

1. *Compute $J(x,y,z)$ by introducing $\mathbf{u} = (u_x, u_y, u_z)$.*

In the Theory of Elasticity the deformations are often small *in the sense that the derivative of $\mathbf{u}$ is small, so we can reject all there* products.

2. *Prove that by this assumption, $J = 1 + \,\mathit{div}\,\mathbf{u}$.*

3. *Finally, prove that the divergence is the relative increase of the volume corresponding to the deformation.*

A Transformation of space integrals.

D Calculate the first approximation of the Jacobian.

I 1) The transformation is given by

$$\begin{aligned} \tilde{x}_1 &= x + u_x(\mathbf{x}), \\ \tilde{x}_2 &= y + u_y(\mathbf{x}), \qquad \mathbf{x} = (x,y,z), \\ \tilde{x}_3 &= z + u_z(\mathbf{x}), \end{aligned}$$

hence

$$\begin{aligned} J(x,y,z) &= \begin{vmatrix} 1 + \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\[2mm] \dfrac{\partial u_x}{\partial y} & 1 + \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\[2mm] \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & 1 + \dfrac{\partial u_z}{\partial z} \end{vmatrix} \\ &= 1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \text{ products of higher order.} \end{aligned}$$

2) If we remove all products of higher order, then we get

$$J = 1 + \operatorname{div} \mathbf{u}.$$

3) The geometrical interpretation of J is given by

$$\mathrm{d}\tilde{\Omega} = |J(\mathbf{x})| \, \mathrm{d}\Omega,$$

where $\mathrm{d}\tilde{\Omega}$ and $\mathrm{d}\Omega$ are considered as infinitesimal volumes corresponding to each other. This may possibly be clarified by

$$\Delta\tilde{\Omega} \approx |J(\mathbf{x})| \, \Delta\Omega.$$

Assume that div $\mathbf{u}$ is small, so higher order terms can be rejected. Then

$$J = 1 + \operatorname{div} \mathbf{u} > 0,$$

and thus

$$\mathrm{d}\tilde{\Omega} = \{1 + \operatorname{div} \mathbf{u}\} \, \mathrm{d}\Omega.$$

The factor $1 + \operatorname{div} \mathbf{u}$ indicates the quotient between the two infinitesimal volumes, so div $\mathbf{u}$ can be interpreted as the relative signed increase of the volume.

Example 29.18 *Let $A \subset \mathbb{R}^3$ be given by*

$$0 \le x, \quad 0 \le y, \quad 0 \le z, \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \le 1.$$

Compute the volume of A and the space integral

$$I = \int_A \exp\left[\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^6\right] \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

by introducing the new variables

$$u = \sqrt{x} + \sqrt{y}, \quad v = \sqrt{x} - \sqrt{y}, \quad w = \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

A Transformation of space integrals.

D Find the inverse transformation and compute the Jacobian before the transformation formula is applied.

I We derive from

$$2\sqrt{x} = u + v, \quad 2\sqrt{y} = u - v, \quad \sqrt{z} = w - u,$$

that

$$x = \frac{1}{4}(u+v)^2, \quad y = \frac{1}{4}(u-v)^2, \quad z = (w-u)^2.$$

Then find the parametric domain B in the (u, v, w)-space.

1) The boundary surface $x = 0$ is mapped into the plane $v = -u$.

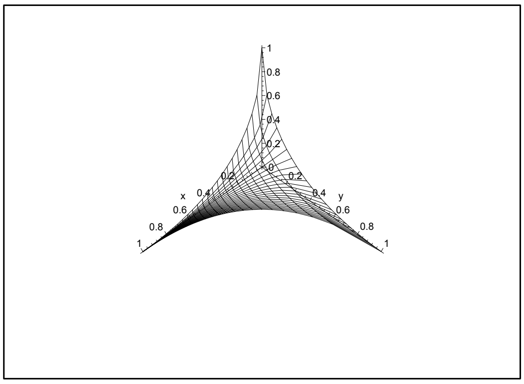

Figure 29.22: The domain A.

2) The boundary surface $y = 0$ is mapped into the plane $v = u$.
3) The boundary surface $z = 0$ is mapped into the plane $w = u$.
4) The boundary surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ is mapped into the plane $w = 1$.

The set A is closed and bounded, and the transformation is continuous. It therefore follows from the second main theorem for continuous functions that A is transformed into the closed and bounded parametric domain

$$B = \{(u,v,w) \mid 0 \le w \le 1,\, 0 \le u \le w,\, -u \le v \le u\}.$$

Then the Jacobian is given by

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(u+v) & \frac{1}{2}(u+v) & 0 \\ \frac{1}{2}(u-v) & -\frac{1}{2}(u-v) & 0 \\ -2(w-u) & 0 & 2(w-u) \end{vmatrix} \\
&= \frac{1}{2}(u+v)\cdot\frac{1}{2}(u-v)\cdot 2(w-u)\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{2}\left(u^2-v^2\right)(w-u)\cdot(-2) = -\left(u^2-v^2\right)(w-u).
\end{aligned}$$

Since $v^2 \le u^2$ and $u \le w$ in B, it follows from the transformation formula that

$$\begin{aligned}
\mathrm{vol}(A) &= \int_A d\Omega = \int_V \left| - \left(-u^2 - v^2\right)(w-u) \right| \, du \, dv \, dw \\
&= \int_0^1 \left\{ \int_0^w \left[\int_{-u}^{u} \left(u^2 - v^2\right)(w-u) \, dv \right] du \right\} dw \\
&= \int_0^1 \left\{ \int_0^{w(w-u)\cdot \frac{4}{3} u^3 \, du} \right\} dw = \int_0^1 \left\{ \frac{4}{3} \int_0^w \left(wu^3 - u^4 \right) du \right\} dw \\
&= \int_0^1 \frac{4}{3} \left[\frac{1}{4} wu^4 - \frac{1}{5} u^5 \right]_{u=0}^{w} dw = \int_0^1 \frac{1}{15} w^5 \, dw = \frac{1}{90},
\end{aligned}$$

and

$$\begin{aligned}
I &= \int_A \exp\left[\left(\sqrt{x} + \sqrt{y} + \sqrt{z} \right)^6 \right] d\Omega \\
&= \int_0^1 \exp\left(w^6\right) \left\{ \int_0^w \left[\int_{-u}^{u} \left(u^2 - v^2\right)(w-u) \, dv \right] du \right\} dw \\
&= \int_0^1 \frac{1}{15} \exp\left(w^6\right) \cdot w^5 \, dw = \frac{1}{90} \int_0^1 e^t \, dt = \frac{e-1}{90},
\end{aligned}$$

where we also found that

$$\int_0^w \left\{ \int_{-u}^{u} \left(u^2 - v^2\right)(w-u) \, dv \right\} du = \frac{1}{15} w^5 .$$

Example 29.19 *Let B be the triangle given by $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Compute the improper plane integral*

$$I = \int_B \exp\left(\frac{x-y}{x+y}\right) \mathrm{d}S$$

by introducing the new variables $(u, v) = (x + y, x - y)$.

A Transformation of an improper plane integral.

D The integrand is not defined at $(x, y) = (0, 0) \in B$. Otherwise, the integrand is positive, so in the worst case we shall only get that the value becomes $+\infty$.

Find x and y expressed by u and v. Find the parametric domain in the (u, v)-plane. Compute the Jacobian. Finally, insert into the transformation formula, check if the singularity has any effect and compute.

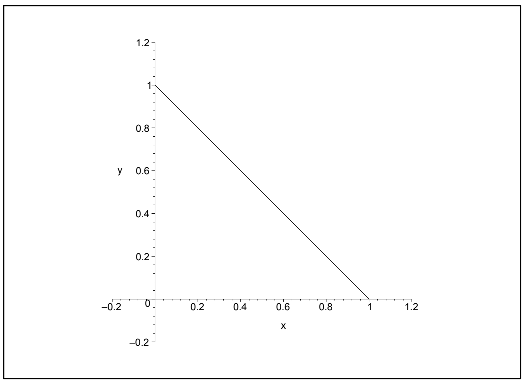

Figure 29.23: The domain of integration B.

I The transformation is continuous with a continuous inverse:

$$\begin{aligned} u &= x + y \quad &\text{and} \quad v &= x - y, \\ x &= \frac{1}{2}(u + v) \quad &\text{and} \quad y &= \frac{1}{2}(u - v). \end{aligned}$$

Furthermore, B is closed and bounded, so by using the second main theorem for continuous functions we conclude that the image, i.e. the new parametric domain A in the (u, v)-plane is also closed and bounded. It therefore suffices to find the images of the boundary curves.

1) $x = 0$ is mapped into $u + v = 0$, i.e. into $v = -u$.
2) $y = 0$ is mapped into $u - v = 0$, i.e. into $v = u$.
3) $x + y = 1$ is mapped into $u = 1$.

It follows that A in the (u, v)-plane is the triangle which is defined by these three lines, hence

$$A = \{(u, v) \mid 0 \leq u \leq 1, \, -u \leq v \leq u\}.$$

Then compute the Jacobian,

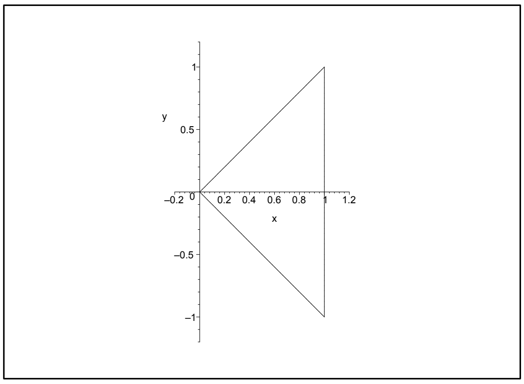

Figure 29.24: The parametric domain A in the (u,v)-plane.

$$\frac{\partial(x,y)}{\partial(u,v)}=\left|\begin{array}{cc}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v}\end{array}\right|=\left|\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\[2mm] \frac{1}{2} & -\frac{1}{2}\end{array}\right|=-\frac{1}{2}.$$

Finally, by putting into the transformation formula where we also have in mind that the integral isan improper one of a positive integrand:

$$\begin{aligned}I &= \int_B \exp\left(\frac{x-y}{x+y}\right)\,\mathrm{d}S=\int_A \exp\left(\frac{v}{u}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{1}{2}\int_0^1\left\{\int_{-u}^{u}\exp\left(\frac{v}{u}\right)\,\mathrm{d}v\right\}\,\mathrm{d}u=\frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1\left\{\int_{-u}^{u}\exp\left(\frac{v}{u}\right)\,\mathrm{d}v\right\}\,\mathrm{d}u\\ &= \frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1 u\left[\exp\left(\frac{v}{u}\right)\right]_{v=-u}^{u}\,\mathrm{d}u=\frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1 u\left(e-e^{-1}\right)\,\mathrm{d}u\\ &= \sinh 1\int_0^1 u\,du=\frac{\sinh 1}{2}.\end{aligned}$$

Example 29.20 *Let A be the tetrahedron which is bounded by the four planes of the equations*

$$x+y+z=0, \quad x+y-z=0, \quad x-y-z=0, \quad 2x-z=1.$$

Calculate the space integral

$$I=\int_A (x+y+z)(x+y-z)(x-y-z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

by introducing the new variables

$$u=x+y+z, \quad v=x+y-z, \quad w=x-y-z.$$

A Transformation of a space integral.

D Find x, y, z expressed by u, v, w. Then find the parametric domain B in the (u,v,w)-space which is uniquely mapped onto A. Compute the Jacobian, and finally, apply the transformation formula.

I It follows from

$$u=x+y+z, \quad v=x+y-z, \quad w=x-y-z,$$

that

$$u+w=2x, \qquad \text{i.e.} \quad x=\frac{1}{2}\,(u+w).$$

Then

$$u-v=2z, \qquad \text{i.e.} \quad z=\frac{1}{2}\,(u-v)$$

and

$$v-w=2y, \qquad \text{i.e.} \quad y=\frac{1}{2}\,(v-w).$$

Summarizing,

$$\begin{array}{lll} u=x+y+z, & v=x+y-z, & w=x-y-z, \\ x=\frac{1}{2}\,(u+w), & y=\frac{1}{2}\,(v-w), & z=\frac{1}{2}\,(u-v), \end{array}$$

and the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}=\begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix}=\frac{1}{8}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}=\frac{1}{8}\,(-1-1)=-\frac{1}{4}.$$

We shall find the images of the boundary surfaces of the tetrahedron:

1) $x+y+z=0$ is mapped into $u=0$.
2) $x+y-z=0$ is mapped into $v=0$.
3) $x-y-z=0$ is mapped into $w=0$.
4) $2x-z=1$, i.e. $2=4x-2z$, is mapped into

$$2=2u+2w-u+v=u+v+2w, \quad \text{i.e.} \quad u+v+2w=2.$$

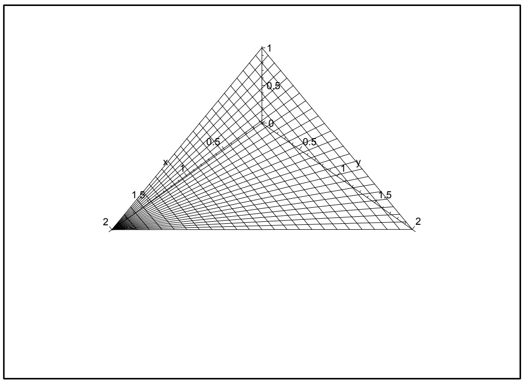

Figure 29.25: The transformed parametric domain B.

The inverse transformation is continuous, and A is closed and bounded. Hence, A is transformed into a new tetrahedron B as indicated on the figure. Note that B is cut at the height $w \in [0,1[$ in the triangle

$$B(w) = \{(u,v) \mid u \geq 0,\, v \geq 0,\, u+v \leq 2(1-w)\}.$$

This can be exploited in the calculation of the transformed integral by the method of slicing.

According to the transformation theorem,

$$\begin{aligned} I &= \int_A (x+y+z)(x+y-z)(x-y-z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_B uvw \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \frac{1}{4}\int_0^1 w\left\{\int_{B(w)} uv\,\mathrm{d}S\right\}\mathrm{d}w. \end{aligned}$$

Then calculate the integral over the slice at height w,

$$\begin{aligned} \int_{B(w)} uv\,\mathrm{d}S &= \int_0^{2(1-w)} u\left\{\int_0^{2(1-w)-u} v\,\mathrm{d}v\right\}\mathrm{d}u = \int_0^{2(1-w)} u\cdot\frac{1}{2}\{2(1-w)-u\}^2\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^{2(1-w)} \left\{4(1-w)^2u - 4(1-w)u^2 + u^3\right\}\,\mathrm{d}u \\ &= \frac{1}{2}\left[2(1-w)^2u^2 - \frac{4}{3}(1-w)u^3 + \frac{1}{4}u^4\right]_0^{2(1-w)} \\ &= \frac{1}{2}\left\{2(1-w)^2\cdot 4(1-w)^2 - \frac{4}{3}(1-w)\cdot 8(1-w)^3 + \frac{1}{4}\cdot 16(1-w)^4\right\} \\ &= \frac{1}{2}(1-w)^4\left\{8 - \frac{32}{3} + 4\right\} = \frac{1}{2}\cdot\frac{4}{3}(1-w)^4 = \frac{2}{3}(w-1)^4. \end{aligned}$$

Finally, by insertion,

$$\begin{aligned} I &= \frac{1}{4}\int_0^1 w\left\{\int_{B(w)} uv\,\mathrm{d}S\right\}\mathrm{d}w = \frac{1}{4}\cdot\frac{2}{3}\int_0^1 w(w-1)^4\,\mathrm{d}w \\ &= \int 16\int_0^1 \left\{(w-1)^5 + (w-1)^4\right\}\,\mathrm{d}w = \frac{1}{6}\left[\frac{1}{6}(w-1)^6 + \frac{1}{5}(w-1)^5\right]_0^1 \\ &= \frac{1}{6}\left\{-\frac{1}{6} - \frac{1}{5}(-1)\right\} = \frac{1}{6}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{180}. \end{aligned}$$

Example 29.21 *Let K denote the closed ball of centrum $(1,1,1)$ and radius $\sqrt{3}$. We construct a subset $A \subset K$ by only keeping those points from K in A, which furthermore satisfy $r \geq 1$ and lie in the first octant.*
Calculate the space integral

$$I = \int_A \frac{1}{r^6}\,\mathrm{d}\Omega$$

by introducing the new variables

$$u = \frac{x}{r^2}, \qquad v = \frac{y}{r^2}, \qquad w = \frac{z}{r^2}.$$

A Transformation of a space integral. This is the 'simplest" non-trivial example in the three dimensional space. We shall see that even in this case the computations grow very big.

D First find A, and then the parametric domain D of the variables (u,v,w). Compute the Jacobian, and finally also the transformed integral.

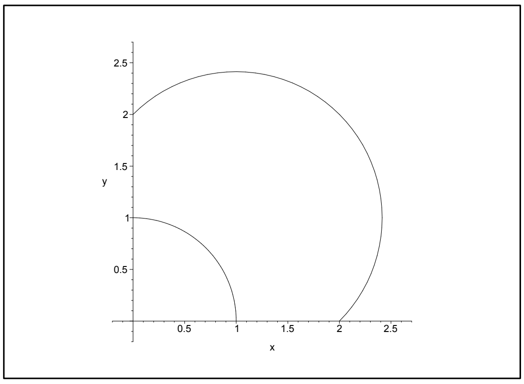

Figure 29.26: The boundary surface of A in each of the three planes $x=0$, $y=0$, or $z=0$.

I The set A is described by

$$A = \left\{(x,y,z) \mid (x-1)^2+(y-)^2+(z-1)^2 \leq 3, x^2+y^2+z^2 \geq 1, x \geq 0, y \geq 0, z \geq 0\right\}.$$

The boundary surface in each of the planes $x=0$, $y=0$ and $z=0$ is indicated on the figure.

Then check the image in the (u,v,w)- space of each of the boundary surfaces in the (x,y,z)-space.

Clearly, the boundary surface $x^2+y^2+z^2=1$ is mapped into $u^2+v^2+z^2=1$, and they both lie in the first octant.

Then check the transformation of the boundary surface

$$(x-1)^2+(y-1)^2+(z-1)^2 = 3.$$

If we put

$$R^2 = u^2+v^2+w^2 = \frac{x^2}{r^4}+\frac{y^2}{r^4}+\frac{z^2}{r^4} = \frac{1}{r^2},$$

then

$$x = u\cdot r^2 = \frac{u}{R^2}, \qquad y = \frac{v}{R^2}, \qquad z = \frac{w}{R^2},$$

hence by insertion

$$(u-R^2)^2+(c-R^2)^2+(w-R^2)^2 = 3R^4,$$

and thus by a computation

$$\begin{aligned} 3R^4 &= u^2-2uR^2+R^4+v^2-2vR^2+w^2-2wR^2+R^4 \\ &= (u^2+v^2+w^2)-2(u+v+w)R^2+3R^4 \\ &= R^2-2(u+v+w)R^2+3R^4 = 3R^4+R^2\{1-2(u+v+w)\}. \end{aligned}$$

Now, $R^2 = u^2+v^2+w^2 = \dfrac{1}{r^2} > 0$, to this is reduced to the equation of a plane surface in the first octant,

$$u+v+w = \frac{1}{2}.$$

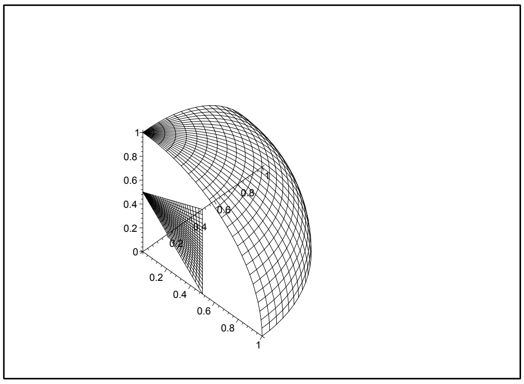

Figure 29.27: The domain D lies in the first octant between the two surfaces.

We conclude that D is that part of the closed first octant, which also lies between the plane $u+v+w = \dfrac{1}{2}$ and the sphere $u^2+v^2+w^2 = 1$.

Since we have

$$u+v+w = R\{\sin\theta(\cos\varphi+\sin\varphi)+\cos\theta\}$$

in spherical coordinates

$$u = R\,\sin\theta\cos\varphi, \quad v = R\,\sin\theta\sin\varphi, \quad w = R\cos\theta,$$

we get the following description of D in spherical coordinates

$$D=\left\{(R,\varphi,\theta)\mid[2\{\sin\theta(\cos\varphi+\sin\varphi)+\cos\theta\}]^{-1}\leq R\leq 1,0\leq\varphi,\theta\leq\frac{\pi}{2}\right\}.$$

Then calculate the Jacobian $\dfrac{\partial(x,y,x)}{\partial(u,v,w)}$, where we use that

$$(x,y,z)=\left(\frac{u}{R^2},\frac{v}{R^2},\frac{w}{R^2}\right).$$

First note that e.g.

$$\frac{\partial}{\partial u}\left(\frac{1}{R^2}\right)=-\frac{1}{R^4}\cdot\frac{\partial R^2}{\partial u}=-\frac{2u}{R^4},$$

and similarly of symmetric reasons,

$$\frac{\partial}{\partial v}\left(\frac{1}{R^2}\right)=-\frac{2v}{R^4}\quad\text{and}\quad\frac{\partial}{\partial w}\left(\frac{1}{R^2}\right)=-\frac{2w}{R^4},$$

Then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{R^2} - \frac{2u^2}{R^4} & -\frac{2uv}{R^4} & -\frac{2uw}{R^4} \\ -\frac{2uv}{R^4} & \frac{1}{r^2} - \frac{2v^2}{R^4} & -\frac{2vw}{R^4} \\ -\frac{2uw}{R^4} & -\frac{2vw}{R^4} & \frac{1}{R^2} - \frac{2w^2}{R^4} \end{vmatrix}$$

$$= \frac{1}{R^{12}} \begin{vmatrix} -u^2+v^2+w^2 & -2uv & -2uw \\ -2uv & u^2-v^2+w^2 & -2vw \\ -2uw & -2vw & u^2+v^2-w^2 \end{vmatrix}$$

$$= \frac{1}{R^{12}} \{(R^2-2u^2)(R^2-2v^2)(R^2-2w^2) - 8u^2v^2w^2 - 8u^2v^2w^2$$
$$-4u^2w^2(R^2-2v^2) - 4v^2w^2(R^2-2u^2) - 4u^2v^2(R^2-2w^2)\}$$
$$= \frac{1}{R^{12}} \{R^6 - 2(u^2+v^2+w^2)R^4 + 4R^2(u^2v^2+u^2w^2+v^2w^2) - 24u^2v^2w^2$$
$$-4R^2(u^2w^2+v^2w^2+u^2v^2) + 8u^2v^2w^2 + 8u^2v^2w^2 + 8u^2v^2w^2\}$$
$$= \frac{1}{R^{12}}\{-R^6\} = -\frac{1}{R^6}.$$

Finally, we get by the transformation theorem and a consideration of a volume that

$$I = \int_A \frac{1}{r^6}\, d\Omega = \int_D R^6 \cdot \frac{1}{R^6}\, d\omega = \int_D d\omega = \mathrm{vol}(D)$$
$$= \frac{1}{8} \cdot \frac{4\pi}{3} - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{\pi}{6} - \frac{1}{48},$$

in which the slicing method is latently applied.

30 Improper integrals

30.1 Introduction

In some cases it is possible to extend the various forms of integrals considered in the previous volumes to situations which are not covered by the theorems already given. We considered earlier (abstract) integrals symbolized by

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu,$$

where A is closed and bounded, and the function $f : A \to \mathbb{R}$ is continuous. We shall in this chapter investigate what can be done if these conditions above are not all fulfilled.

Let us start with a useful technical trick, which will be very important in the discussion below. We shall split the function f under consideration into its positive part f_+ and its negative part f_-, and then continue by only discussing the case where f is nonnegative.

Given any function $f : A \to \mathbb{R}$, we define f_+ and f_- in the following way,

$$f_+(\mathbf{x}) := \begin{cases} f(\mathbf{x}), & \text{if } f(\mathbf{x}) > 0, \\ 0, & \text{if } f(\mathbf{x}) =\leq 0, \end{cases} \qquad f_-(\mathbf{x}) := \begin{cases} -f(\mathbf{x}), & \text{if } f(\mathbf{x}) < 0, \\ 0, & \text{if } f(\mathbf{x}) \geq 0. \end{cases}$$

Then clearly both f_+ and f_- are *nonnegative*, and

$$f(\mathbf{x}) = f_+(\mathbf{x}) - f_-(\mathbf{x}) \quad \text{and} \quad f_+(\mathbf{x}) \cdot f_-(\mathbf{x}) = 0 \qquad \text{for all } \mathbf{x} \in A.$$

Furthermore, the splitting of f, satisfying the two conditions above, is unique.

Now, if we can prescribe (finite) values of both $\int_A f_+(\mathbf{x})\,\mathrm{d}\mu$ and $\int_A f_-(\mathbf{x})\,\mathrm{d}\mu$, then we simply define the value of the improper integral $\int_A f(\mathbf{x})\,\mathrm{d}\mu$ by putting

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = \int_A f_+(\mathbf{x})\,\mathrm{d}\mu - \int_A f_-(\mathbf{x})\,\mathrm{d}\mu.$$

This means that we in the following only shall consider *nonnegative integrands*, $f \geq 0$, when we extend the plane, space, line or surface integrals to the case of improper integrals, where either f is not continuous, or A is either not bounded or not closed.

Once we have restricted the investigation to nonnegative functions $f \geq 0$, the idea of defining the improper integral $\int_A f(\mathbf{x})\,\mathrm{d}\mu$ goes as follows. We choose a sequence of *nested closed and bounded sets*, i.e.

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset A,$$

such that the given nonnegative function f is continuous on every A_n, and such that $A_n \to A$ in some sense for $n \to +\infty$. Since f is continuous on every A_n, and A_n is closed and bounded, the integral of f over each A_n does indeed exist, and since $f \geq 0$ is nonnegative, and the sequence A_n is nested, i.e. $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$, we conclude that the sequence

$$\int_{A_1} f(\mathbf{x})\,\mathrm{d}\mu \leq \int_{A_2} f(\mathbf{x})\,\mathrm{d}\mu \leq \cdots \leq \int_{A_n} f(\mathbf{x})\,\mathrm{d}\mu \leq \cdots,$$

of real numbers is (weakly) increasing, hence it is either convergent, or tending towards $+\infty$. This is the reason for only considering nonnegative functions.

It is not hard to see that any nested increasing sequence of closed and bounded subsets of A, on which $f \geq 0$ is continuous, will do. We still have to explain, what is meant by $A_n \to A$ for $n \to +\infty$.

If

$$\bigcup_{n=1}^{+\infty} A_n = A,$$

then the immediate definition of the improper integral of f over A becomes

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = \lim_{n\to+\infty} \int_{A_n} f(\mathbf{x})\,\mathrm{d}\mu.$$

If the limit is convergent towards the value $I \in \mathbb{R}$, we say that the improper integral is convergent and

$$\int_A f(\mathbf{x})\,\mathrm{d}\mu = I.$$

If the limit is $+\infty$, we say that the improper integral is divergent.

However, measures are only determined modulo null sets, so we may weaken the definition to the requirement that only

$$\bigcup_{n=1}^{+\infty} A_n = A \setminus M, \qquad \text{where } \mu(M) = 0, \quad \text{i.}e. \quad M \text{ is a null set.}$$

Then we can still use the definitions above of the improper integral, because the "missing values" from $f(\mathbf{x})$ over the null set M have by definition only the weight 0, so we only exclude 0.

30.2 Theorems for proper integrals

A further analysis of the improper integrals shows that we can divide those considered here into two types, which may not be disjoint,

1) Unbounded continuous integrand.

2) Unbounded domain of integration.

Only these cases will be relevant for us, although the classification above is not complete. There exist improper integrals, which cannot be broken down to a finite number of cases of the types above, but they will be outside the realm of this chapter.

The proofs of the following theorems have already been sketched in Section 30.1. We shall be satisfied with these sketches and not bother with more precise proofs.

Theorem 30.1 . ***Improper integral with unbounded integrand.*** *Let $A \subset \mathbb{R}^k$ be a bounded set, and let $M \subset A$ be a null set, $\mu(M) = 0$. Assume that*

$$f : A \setminus M \to [0, +\infty[$$

is a nonnegative continuous and unbounded function. Choose an increasing (nested) sequence of sets

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset A,$$

such that

$$M \subset A \setminus A_n \quad \text{for all } n \in \mathbb{N}, \qquad \text{and} \qquad \mu\left(A \setminus A_n\right) \to 0 \quad \text{for } n \to +\infty.$$

Then either

$$\lim_{n \to +\infty} \int_{A_n} f(\mathbf{x})\, \mathrm{d}\mu = +\infty,$$

and we call the improper integral divergent, or,

$$\lim_{n \to +\infty} \int_{A_n} f(\mathbf{x})\, \mathrm{d}\mu = I < +\infty,$$

is convergent, in which case we say that the improper integral is convergent of the value

$$\int_A f(\mathbf{x})\, \mathrm{d}\mu := \lim_{n \to +\infty} \int_{A_n} f(\mathbf{x})\, \mathrm{d}\mu.$$

Theorem 30.2 ***Improper integral with unbounded domain of integration.*** *Assume that $A \subseteq \mathbb{R}^k$ is unbounded, and that $f : A \to \mathbb{R}$ is a nonnegative and continuous function. Choose an increasing (nested) sequence of closed and bounded sets,*

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset A, \qquad \text{such that } \bigcup_{n=1}^{+\infty} A_n = A.$$

then either

$$\lim_{n \to +\infty} \int_{A_n} f(\mathbf{x})\, \mathrm{d}\mu = +\infty,$$

and the improper integral of f over A is said to be divergent, or it is convergent of the value

$$\int_A f(\mathbf{x})\, \mathrm{d}\mu = \lim_{n \to +\infty} \int_{A_n} f(\mathbf{x})\, \mathrm{d}\mu < +\infty.$$

In the applications we usually have to combine these two theorems.

30.3 Procedure for improper integrals; bounded domain

Problem 30.1 *Calculate the integral*

$$\int_A f(\mathbf{x})\,\mathrm{d}S,$$

where the domain A is bounded, and where either A is not closed, or the integrand $f(\mathbf{x})$ is not defined for all $\mathbf{x} \in A$.

We choose in this description for convenience $A \subset \mathbb{R}^2$. The procedure is analogous for any $A \subset \mathbb{R}^k$.

Idea: Approximate $\int_A f(\mathbf{x})\,\mathrm{d}S$ by $\int_{A_n} f(\mathbf{x})\,\mathrm{d}S$, where

1) The function $f(\mathbf{x})$ is continuous for all $x \in A_n$.

2) All sets $A_n \subset A$ are closed and bounded, $n \in \mathbb{N}$.

3) $\mathrm{area}(A \setminus A_n) \to 0$ for $n \to +\infty$.

Remark 30.1 Even if we are approaching the solution by this idea, it is not enough. We shall below supply it with some sufficient conditions in the description of the procedure. $\Diamond$

Procedure.

1) Examine whether $f(\mathbf{x})$ has a continuous extension $F(\mathbf{x})$ to $\overline{A}$. If this is the case, then

$$\int_A f(\mathbf{x})\,\mathrm{d}S = \int_{\overline{A}} F(\mathbf{x})\,\mathrm{d}S,$$

where the right hand side is calculated by well-known methods, and the problem is solved.

2) If $f(\mathbf{x})$ does *not* have a continuous extension to $\overline{A}$, we continue by an *analysis of the sign of* $f(\mathbf{x})$. This step is extremely *important!* Divide A into two subsets

$$A^+ = \{(x,y) \in A \mid f(x,y) \geq 0\} \quad \text{and} \quad A^- = \{(x,y) \in A \mid f(x,y) \leq 0\}.$$

If $f(x,y)$ is continuous where it is defined, then this division is most easily performed by finding the *null curves* i.e. the curves where $f(x,y) = 0$. These curves divide the domain into open sub-domains. Due to the continuity the sign of f is constant in each of these sub-domains, so in order to find the sign one just has to apply f to one point in each sub-domain. The curves where $f(x,y) = 0$ can afterwards be included in any of the two sets A^+ or A^-, and even in both of them, because since the integrand is 0 on these curves they will not contribute the the final result.

3) Let us first consider A^+. If $A^+ = \emptyset$, then go to 4) below. If on the other hand $A^+ \neq \emptyset$, we choose a convenient increasing sequence of *closed and bounded* subsets $A_\varepsilon^+ \subset A^+$, such that

$$\mathrm{area}(A^+ \setminus A_\varepsilon^+) \to 0 \qquad \text{for } \varepsilon \to 0+.$$

Calculate

$$I_\varepsilon^+ = \int_{A_\varepsilon^+} f(x,y)\,\mathrm{d}S \qquad (\geq 0).$$

If $I_\varepsilon^+ \to +\infty$ for $\varepsilon \to 0+$, then the improper integral is *divergent*.

If on the other hand $\lim_{\varepsilon\to 0} I_\varepsilon^+ < +\infty$ (there are only these two possibilities, because I_ε^+ increases, when ε decreases towards 0), then

$$\int_{A^+} f(x,y)\,\mathrm{d}S = \lim_{\varepsilon\to 0+} \int_{A_\varepsilon^+} f(x,y)\,\mathrm{d}S \qquad \geq 0.$$

4) Then consider the other subset A^-. If $A^- = \emptyset$, go to 5).

If on the other hand, $A^- \neq \emptyset$, choose a convenient increasing sequence of *closed and bounded* subsets $A_\varepsilon^- \subset A^-$, such that

$$\operatorname{area}(A^- \setminus A_\varepsilon^-) \to 0 \qquad \text{for } \varepsilon \to 0+.$$

Calculate

$$I_\varepsilon^- = \int_{A_\varepsilon^-} f(x,y)\,\mathrm{d}S \qquad (\leq 0).$$

If $I_\varepsilon^- \to -\infty$ for $\varepsilon \to 0$, then the improper integral is *divergent*.

If instead $\lim_{\varepsilon\to 0} I_\varepsilon^- > -\infty$ (there are only these two possibilities), then

$$\int_{A^-} f(x,y)\,\mathrm{d}S = \lim_{\varepsilon\to 0+} \int_{A_\varepsilon^-} f(x,y)\,\mathrm{d}S \qquad (\leq 0).$$

5) It is only when we have obtained *convergence* in both 3) and 4) that we can conclude that the improper integral is *convergent* with the value

$$\int_A f(x,y)\,\mathrm{d}S = \int_{A^+} f(x,y)\,\mathrm{d}S + \int_{A^-} f(x,y)\,\mathrm{d}S.$$

30.4 Procedure for improper integrals; unbounded domain

Problem 30.2 *Calculate the integral*

$$\int_A f(x,y)\,\mathrm{d}S,$$

where the domain of integration $A \subseteqq \mathbb{R}^2$ *is unbounded.*

Procedure:

1) Examine whether f has a continuous extension to $\overline{A} \setminus \{\infty\}$. If this is not the case, remove suitable bounded neighbourhoods of the "sick" points, and treat each of these neighbourhoods as previously.

   The residual set which we here denote by A is then treated in the following way.

2) The important *analysis of the sign of the integrand.* Divide A into the two sets

$$A^+ = \{(x,y) \in A \mid f(x,y) \geq 0\} \quad \text{and} \quad A^- = \{(x,y) \in A \mid f(x,y) \leq 0\}.$$

   The sets A^+ and A^- are found by analyzing the null curves, i.e. the set defined by the equation $f(x,y) = 0$.

3) Let us first consider A^+. If $A^+ = \emptyset$, then go to 4) in the following.

   If $A^+ \neq \emptyset$, we choose one of the possible methods of trimming the set

   a) Rectangular: $A_n^+ = [-n,n]^2 \cap A^+$, b) Polar: $A_R^+ = \overline{K}(\mathbf{0};R) \cap A^+$,

   and we calculate

   a) Rectangular: $I_n^+ = \displaystyle\int_{A_n^+} f(x,y)\,\mathrm{d}S$, b) Polar: $I_R^+ = \displaystyle\int_{A_R^+} f(x,y)\,\mathrm{d}S$.

   If $I_n^+ \to +\infty$ for $n \to +\infty$, resp. $I_R^+ \to +\infty$ for $R \to +\infty$, then the improper integral is *divergent.*

   Otherwise,

   a) Rectangular: $\int_{A^+} f(x,y)\,\mathrm{d}S = \lim_{n\to+\infty} \int_{A_n^+} f(x,y)\,\mathrm{d}S$.

   b) Polar: $\int_{A^+} f(x,y)\,\mathrm{d}S = \lim_{R\to+\infty} \int_{A_R^+} f(x,y)\,\mathrm{d}S$.

   Then go to 4).

4) Next turn to A^-. If $A^- = \emptyset$, then go to 5).

   If $A^- \neq \emptyset$, choose one of the trimming possibilities

   a) Rectangular: $A_n^- = [-n,n]^2 \cap A^-$, b) Polar: $A_R^- = \overline{K}(\mathbf{0};R) \cap A^-$,

   and calculate

   a) Rectangular: $I_n^- = \displaystyle\int_{A_n^-} f(x,y)\,\mathrm{d}S$, b) Polar: $I_R^- = \displaystyle\int_{A_R^-} f(x,y)\,\mathrm{d}S$.

If $I_n^- \to -\infty$ for $n \to +\infty$, resp. $I_R^- \to -\infty$ for $R \to +\infty$, then the improper integral is *divergent*.

Otherwise,

a) Rectangular: $\int_{A^-} f(x,y)\,\mathrm{d}S = \lim_{n\to+\infty} \int_{A_n^-} f(x,y)\,\mathrm{d}S.$

b) Polar: $\int_{A^-} f(x,y)\,\mathrm{d}S = \lim_{R\to+\infty} \int_{A_R^-} f(x,y)\,\mathrm{d}S.$

5) If we obtain *convergence* in both 3) and 4), then the improper integral is *convergent* with the value

$$\int_A f(x,y)\,\mathrm{d}S = \int_{A^+} f(x,y)\,\mathrm{d}S + \int_{A^-} f(x,y)\,\mathrm{d}S.$$

Remark 30.2 If we put

$$A^0 = \{(x,y) \in A \mid f(x,y) = 0\},$$

it is seen that

$$\int_{A^0} f(x,y)\,\mathrm{d}S = \int_{A^0} 0\,\mathrm{d}S = 0,$$

so it is of no importance whether we include A^0 in our calculations, or not. The contribution from A^0 is always 0. $\diamondsuit$

30.5 Examples of improper integrals

Example 30.1

A. *Let $B = [0,1]^2$ be the unit square. Examine whether the improper plane integral*

$$I = \int_B \frac{1}{y-x-1}\,\mathrm{d}S$$

is convergent. If this is the case, find its value.

D. The domain B is closed and bounded; but the integrand is not defined in all points of B. Cut away open neighbourhoods of the points where the integrand is not defined; note that the integrand does not change sign; calculate the integral over any of the truncated domains and finally go to the limit.

I. The denominator must never be zero, so we have to avoid the line $y = x+1$. This line cuts B at the point $(0,1)$, which must be removed from the domain of integration.

Then we realize that $\dfrac{1}{y-x-1} < 0$ everywhere in $B \setminus \{(0,1)\}$, hence the integrand does not change sign in the part of B, where it is defined. Therefore we shall not further divide the domain according to the positive and the negative part of the integrand.

Our next step is to truncate B, such that the singular point $(0,1)$ does not lie in any of the closed and bounded domains $B(\varepsilon)$, and such that

$$\operatorname{areal}(B \setminus B(\varepsilon)) \to 0 \qquad \text{for } \varepsilon \to 0.$$

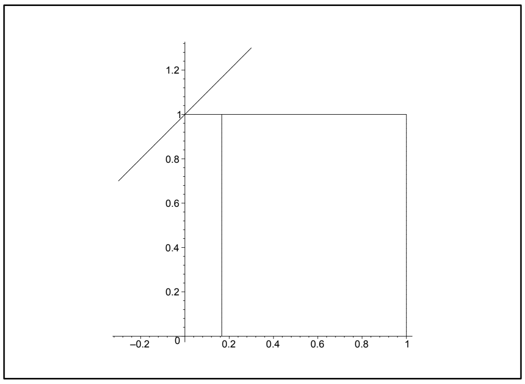

Figure 30.1: The domain B with the oblique singular line $y = x+1$ and the singular point $(0,1)$, and a convenient truncation parallel to the Y-axis.

The most reasonable truncations among many possibilities are given by

$$B(\varepsilon) = [\varepsilon, 1] \times [0,1] \quad \text{and} \quad \tilde{B}(\varepsilon) = [0,1] \times [\varepsilon, 1],$$

for $0 < \varepsilon < 1$. We shall here choose the first one. Then by a reduction,

$$I(\varepsilon) = \int_{B(\varepsilon)} \frac{1}{y-x-1}\, \mathrm{d}S = \int_\varepsilon^1 \left\{ \int_0^1 \frac{1}{y-x-1}\, \mathrm{d}y \right\} \mathrm{d}x.$$

We calculate the inner integral,

$$\int_0^1 \frac{1}{y-x-1}\, \mathrm{d}y = [\ln|y-x-1|]_0^1 = \ln|-x| - \ln|-x-1| = \ln x - \ln(x+1).$$

Then insert the result followed by a partial integration,

$$\begin{aligned} I(\varepsilon) &= \int_\varepsilon^1 \{1 \cdot \ln x - 1 \cdot \ln(x+1)\}\, \mathrm{d}x \\ &= [x \ln x]_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x}\, \mathrm{d}x - [(x+1)\ln(x+1)]_\varepsilon^1 + \int_\varepsilon^1 (x+1) \cdot \frac{1}{x+1}\, \mathrm{d}x. \end{aligned}$$

The clever trick is here to choose $\int \mathrm{d}x = x$ in the first partial integration, and $\int \mathrm{d}x = x+1$ in the second one. In fact, the antiderivatives are only determined modulo a constant. By this trick we get

$$\begin{aligned} I(\varepsilon) &= 1 \cdot \ln 1 - \varepsilon \ln \varepsilon - 2 \ln 2 + (1+\varepsilon)\ln(1+\varepsilon) \\ &= -2 \ln 2 - \varepsilon \cdot \ln \varepsilon = (1+\varepsilon)\ln(1+\varepsilon). \end{aligned}$$

Since area $B \setminus B(\varepsilon) = \varepsilon \to 0$ for $\varepsilon \to 0+$, this is the right limit. It follows from the magnitude of functions that

$$\varepsilon \ln \varepsilon \to 0 \qquad \text{for } \varepsilon \to 0+,$$

and since

$$(1+\varepsilon)\cdot\ln(1+\varepsilon)\to 1\cdot\ln 1 = 0 \qquad \text{for } \varepsilon\to 0+,$$

we conclude that

$$\begin{aligned} I &= \lim_{\varepsilon\to 0+} I(\varepsilon) \\ &= -2\,\ln 2 - \lim_{\varepsilon\to 0+} \varepsilon\cdot\ln\varepsilon + \lim_{\varepsilon\to 0+}(1+\varepsilon)\cdot\ln(1+\varepsilon) \\ &= -2\,\ln 2, \end{aligned}$$

i.e. the improper integral is *convergent* and its value is

$$I = \int_B \frac{1}{y-x-1}\,\mathrm{d}S = -2\,\ln 2 < 0.$$

C. *A very weak control.* The integrand is negative, where it is defined. Hence the result should also be negative, which is seen to be true. (Note that this is only catching errors, where we end up with a *positive* result. Negative wrong results cannot be traced in this way). ◊

Example 30.2

A. *Let $\mathcal{F}$ be the half-sphere $\mathcal{F}$ in the upper half plane of radius $a > 0$, i.e. in spherical coordinates,*

$$r = a, \qquad 0 \le \theta \le \frac{\pi}{2}, \qquad 0 \le \varphi \le 2\pi.$$

Examine whether the improper surface integral

$$I = \int_{\mathcal{F}} \frac{1}{z}\,\mathrm{d}S$$

is convergent or divergent.

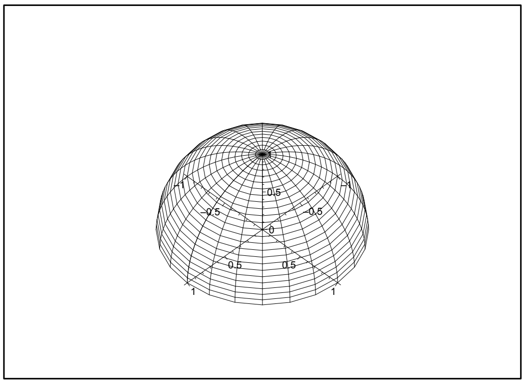

Figure 30.2: The surface $\mathcal{F}$ for $a = 1$.

D. *Surface integrals* can also be improper!
In the formulation of the task there is a hint of using *spherical* coordinates; though there exist also quite reasonable variants of solutions in *semi-polar* and *rectangular* coordinates.

The points where the integrand is not defined lie on the circle

$$\{(x, y, 0) \mid x^2 + y^2 = a^2\}$$

in the (x, y)-plane. Hence by the truncations we shall stay away from the (x, y)-plane. On the residual part of the surface we see that the integrand $\dfrac{1}{z} > 0$, so no further division of the domain is needed concerning the sign of the integrand.

I 1. *Spherical coordinates.* In this case we truncate $\mathcal{F}$ to $\mathcal{F}(\varepsilon)$ by

$$\mathcal{F}(\varepsilon): \quad r = a, \quad 0 \le \theta \le \frac{\pi}{2} - \varepsilon, \quad 0 \le \varphi \le 2\pi, \quad \text{where } 0 < \varepsilon < \frac{\pi}{2}.$$

A geometrical consideration gives that

$$\operatorname{area}(\mathcal{F} \setminus \mathcal{F}(\varepsilon)) \to 0 \qquad \text{for } \varepsilon \to 0+,$$

and as mentioned in **D**. the integrand is positive, so we shall not divide the domain further.

By a reduction of the ordinary *surface integral* over $\mathcal{F}(\varepsilon)$ we get

$$\begin{aligned} I(\varepsilon) &= \int_{\mathcal{F}(\varepsilon)} \frac{1}{z}\,\mathrm{d}S = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}-\varepsilon} \frac{1}{a\,\cos\theta} \cdot a^2 \sin\theta\,\mathrm{d}\theta \right\} \mathrm{d}\varphi \\ &= a \cdot 2\pi \int_0^{\frac{\pi}{2}-\varepsilon} \frac{1}{\cos\theta} \cdot \sin\theta\,\mathrm{d}\theta = 2\pi a\,[-\ln\cos\theta]_0^{\frac{\pi}{2}-\varepsilon} \\ &= 2\pi a\left\{\ln\cos 0 - \ln\cos\left(\frac{\pi}{2}-\varepsilon\right)\right\} \\ &= -2\pi a\,\ln\left\{\cos\frac{\pi}{2}\cdot\cos\varepsilon + \sin\frac{\pi}{2}\cdot\sin\varepsilon\right\} \\ &= -2\pi a\,\ln\{0+\sin\varepsilon\} = -2\pi a\,\ln\sin\varepsilon. \end{aligned}$$

Since $\sin\varepsilon \to 0+$ for $\varepsilon \to 0+$, we have $\ln\sin\varepsilon \to -\infty$ for $\varepsilon \to 0+$, i.e. $I(\varepsilon) \to -(-\infty) = +\infty$ for $\varepsilon \to 0+$, and the improper surface integral is *divergent*.

I 2. *Semi-polar coordinates*. Here the surface is described by

$$\mathcal{F}: \quad 0 \le z \le a \quad \text{and} \quad \varrho = \sqrt{a^2 - z^2},$$

i.e. $\varrho^2 + z^2 = a^2$, $\varrho \ge 0$. The surface is truncated in the following way,

$$\mathcal{F}(\varepsilon): \quad \varepsilon \le z \le a, \quad \varrho = \sqrt{a^2-z^2}, \qquad \text{where } 0 < \varepsilon < a,$$

and it follows geometrically that area $(\mathcal{F} \setminus \mathcal{F}(\varepsilon)) \to 0$ for $\varepsilon \to 0+$.

The surface integral over $\mathcal{F}(\varepsilon)$ is reduced in the following

$$I(\varepsilon) = \int_{\mathcal{F}(\varepsilon)} \frac{1}{z}\,\mathrm{d}S = \int_\varepsilon^a \frac{1}{z}\left\{\int_{B(z)} \mathrm{d}s\right\}\mathrm{d}z = \int_\varepsilon^a \frac{1}{z}\,\text{length}\{B(z)\}\,\mathrm{d}z,$$

because $\mathcal{F}(\varepsilon)$ is cut at height $z \in [\varepsilon, a]$ in a circle $B(z)$ of radius $\varrho = \sqrt{a^2-z^2}$, such that the inner integral is the length of $B(z)$, i.e.

$$2\pi\,\varrho = 2\pi\sqrt{a^2-z^2}.$$

Then we get by insertion

$$I(\varepsilon) = 2\pi \int_\varepsilon^a \frac{1}{z}\sqrt{a^2-z^2}\,\mathrm{d}z.$$

When z is small, then $\sqrt{a^2-z^2} \approx a$, hence the integrand is $\approx \dfrac{a}{z}$. This function cannot be integrated from 0, so this gives a hint that we may have divergence. Let us prove this.

The integrand is positive everywhere. If $0 < z \le \dfrac{\sqrt{3}}{2}\,a$, then

$$\sqrt{a^2-z^2} \ge \sqrt{a^2 - \frac{3}{4}a^2} = \frac{a}{2}.$$

We then have for $0 < \varepsilon < \dfrac{\sqrt{3}}{2}\, a$ the following estimates

$$\begin{aligned}
I(\varepsilon) &= 2\pi \int_{\varepsilon}^{a} \frac{1}{z}\sqrt{a^2 - z^2}\,\mathrm{d}z \geq 2\pi \int_{\varepsilon}^{\frac{\sqrt{3}}{2}\,a} \frac{1}{z}\sqrt{a^2 - z^2}\,\mathrm{d}z \\
&\geq 2\pi \int_{\varepsilon}^{\frac{\sqrt{3}}{2}\,a} \frac{a}{2} \cdot \frac{1}{z}\,\mathrm{d}z = \pi a \int_{\varepsilon}^{\frac{\sqrt{3}}{2}\,a} \frac{\mathrm{d}z}{z} \\
&= \pi\, a \left[\ln z\right]_{\varepsilon}^{\frac{\sqrt{3}}{2}\,a} = \pi a \left\{ \ln\left(\frac{\sqrt{3}}{2}\,a\right) - \ln \varepsilon \right\} \to +\infty \quad \text{for } \varepsilon \to \infty,
\end{aligned}$$

and we conclude that the improper surface integral is *divergent*.

I 3. *Rectangular coordinates.* In this case we consider $\mathcal{F}$ as the graph of the function

$$z = f(x,y) = \sqrt{a^2 - x^2 - y^2}, \qquad (x,y) \in E,$$

where the parametric domain is

$$E = \{(x,y) \mid x^2 + y^2 \leq a^2\}.$$

The natural truncation of the domain is here

$$E_{\varepsilon} = \{(x,y) \mid x^2 + y^2 \leq (a-\varepsilon)^2\}, \qquad 0 < \varepsilon < a,$$

and it is obvious that $\operatorname{area}(E \setminus E_{\varepsilon}) \to 0$ for $\varepsilon \to 0+$.

The *weight function* is in the case of a graph given by

$$\begin{aligned}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{y}{\sqrt{a^2-x^2-y^2}}\right)^2} \\ &= \frac{a}{\sqrt{a^2-x^2-y^2}}.\end{aligned}$$

Then we get by the reduction formula,

$$\begin{aligned} I(\varepsilon) &= \int_{\mathcal{F}(\varepsilon)} \frac{1}{z}\,\mathrm{d}S = \int_{E_\varepsilon} \frac{1}{\sqrt{a^2-x^2-y^2}} \cdot \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= a\int_{E_\varepsilon} \frac{1}{a^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = a\int_0^{2\pi}\left\{\int_0^{a-\varepsilon} \frac{1}{a^2-\varrho^2}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\ &= a\cdot 2\pi\int_0^{(a-\varepsilon)^2} \frac{1}{a^2-t}\cdot\frac{1}{2}\,\mathrm{d}t = \pi\,a\left[-\ln\left|(a^2-t)\right|\right]_0^{(a-\varepsilon)^2} \\ &= \pi\,a\left\{\ln a^2 - \ln\{a^2-(a-\varepsilon)^2\}\right\} \\ &= \pi\,a\left\{2\,\ln a - \ln\{(2a-\varepsilon)\varepsilon\}\right\} \\ &= 2\pi\,a\,\ln a - \pi\,a\,\ln(2a-\varepsilon) - \pi\,a\,\ln\varepsilon \\ &\to 2\pi\,a\,\ln a - \pi\,a\,\ln 2a - (-\infty) = +\infty \qquad \text{for } \varepsilon\to 0+, \end{aligned}$$

and we conclude again that the improper surface integral is *divergent*.

Remark 30.3 In this case we could use all the three classical coordinate systems. Note that the three proofs are totally different in their arguments. $\Diamond$

Example 30.3

A. *Let B be the disc given by $x^2+y^2\le a^2$. Find all values of $\alpha\in\mathbb{R}$, for which the (proper or improper) plane integral*

$$J(\alpha) = \int_B (a^2-y^2-x^2)^\alpha\,\mathrm{d}S$$

is convergent. In case of convergence, find its value.

This example is of the same type as

$$\int_0^a t^\alpha\,\mathrm{d}t = \begin{cases} \dfrac{1}{\alpha+1}\,a^{\alpha+1} & \text{for } \alpha > -1, \\[2mm] \text{divergent} & \text{for } \alpha\le -1, \end{cases}$$

and

$$\int_a^{+\infty} t^\alpha\,\mathrm{d}t = \begin{cases} \text{divergent} & \text{for } \alpha\ge -1, \\[2mm] \dfrac{1}{|\alpha+1|}\cdot\dfrac{1}{a^{|\alpha+1|}} & \text{for } \alpha < -1, \end{cases}$$

known from the *Theory of Functions in One Variable.*

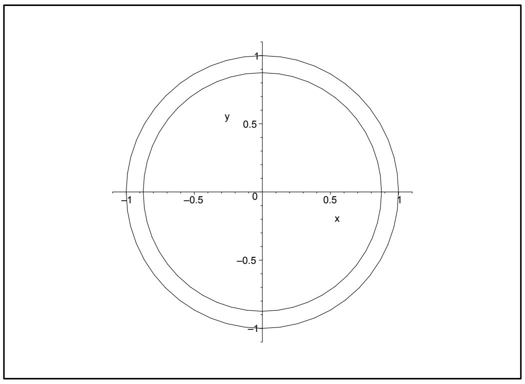

Figure 30.3: The disc B and the truncation $B(p)$ for $a = 1$.

D. *Dimensional considerations* are here extremely useful. In fact, x, $y \sim a$ and $\int_B \cdots dS \sim a^2$, hence

$$J(\alpha) \sim a^{2\alpha} \cdot a^2 = a^{2(\alpha+1)}.$$

In order to get convergence we must have

$$a^{2(\alpha+1)} \to 0 \qquad \text{for } a \to 0+,$$

i.e. $\alpha + 1 > 1$, or $\alpha > -1$. Since

$$a^{2(\alpha+1)} \to +\infty \qquad \text{for } a \to 0+, \text{ when } \alpha < -1,$$

we may expect divergence in this case. (A rough argument is the following: The integral of a positive term tends to $+\infty$, when the domain is shrunk. This is only possible, when we *start* with the value $+\infty$, i.e. with divergence).

The integral is proper, when $\alpha \geq 0$, and it is improper when $\alpha < 0$. When the integrand is defined, it is positive, so we shall not bother with an extra division of the domain according to the positive and negative part of the integrand.

If $\alpha < 0$, then the integrand is not defined on the boundary (the circle) $x^2 + y^2 = a^2$.

The considerations above indicate that the value $\alpha = -1$ divides convergence and divergence. For that reason we start by first examining this case.

I. When we truncate, it is of no importance whether the integral is proper or improper. Since the integrand is positive (of fixed sign) we shall not make any further division of the domain. We truncate by the definition

$$B(p) = \{(x, y) \mid x^2 + y^2 \leq (p\,a)^2\}, \qquad 0 < p < 1,$$

and we note that area$\{B \setminus B(p)\} \to 0$ for $p \to 1-$.

1) If $\alpha = -1$, we get by using *polar* coordinates in $B(p)$ that

$$\begin{aligned}
J(-1;p) &= \int_{B(p)} \frac{1}{a^2-x^2-y^2}\,\mathrm{d}S = \int_0^{2\pi}\left\{\int_0^{p\,a} \frac{1}{a^2-\varrho^2}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= 2\pi\int_0^{(p\,a)^2} \frac{1}{a^2-t^2}\cdot\frac{1}{2}\,\mathrm{d}t = \pi\left[-\ln|a^2-t|\right]_0^{(p\,a)^2} \\
&= \pi\left\{\ln a^2 - \ln\left(a^2-p^2a^2\right)\right\} = \pi\,\ln\left\{\frac{a^2}{a^2(1-p^2)}\right\} \\
&= \pi\,\ln\left(\frac{1}{1-p^2}\right) \to +\infty \qquad \text{for } p\to -1.
\end{aligned}$$

We therefore conclude that

$$J(-1) = \int_B \frac{dS}{a^2-x^2-y^2} = \lim_{p\to 1-} J(-1;p) = +\infty,$$

is divergent.

2) If $\alpha < -1$, i.e. $\alpha+1<0$, then we can use 1) in the following rearrangements and estimates,

$$\begin{aligned}
\left(a^2-x^2-y^2\right)^{\alpha} &= \left(a^2-x^2-y^2\right)^{\alpha+1}\cdot\frac{1}{a^2-x^2-y^2} \\
&= a^{2(\alpha+1)}\cdot\left\{1-\frac{x^2+y^2}{a^2}\right\}^{\alpha+1}\cdot\frac{1}{a^2-x^2-y^2} \\
&= a^{2(\alpha+1)}\cdot\frac{1}{\left\{1-\dfrac{x^2+y^2}{a^2}\right\}^{|\alpha+1|}}\cdot\frac{1}{a^2-x^2-y^2} \\
&\geq a^{2(\alpha+1)}\cdot\frac{1}{a^2-x^2-y^2},
\end{aligned}$$

because

$$\left\{1-\frac{x^2+y^2}{a^2}\right\}^{|\alpha+1|} \leq 1.$$

Then it follows from 1) that

$$\begin{aligned}
J(\alpha;p) &= \int_{B(p)} \left(a^2-x^2-y^2\right)^{\alpha}\,\mathrm{d}S \geq a^{2(\alpha+1)}\int_{B(p)}\frac{1}{a^2-x^2-y^2}\,\mathrm{d}S \\
&= a^{2(\alpha+1)}\,J(-1;p)\to+\infty \qquad \text{for } p\to 1-,
\end{aligned}$$

i.e. $J(\alpha;p)\to+\infty$ for $p\to 1-$, and we have got *divergence* for $\alpha<-1$, and hence also for $\alpha\leq -1$.

3) If $\alpha > -1$, i.e. $\alpha+1>0$, we get by using *polar* coordinates in $B(p)$ that

$$\begin{aligned}
J(\alpha;p) &= \int_{B(p)} \left(a^2-x^2-y^2\right)^{\alpha}\,\mathrm{d}S = \int_0^{2\pi}\left\{\int_{black0}^{p\,a}\left(a^2-\varrho^2\right)^{\alpha}\cdot\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= 2\pi\int_{a^2}^{a^2(1-p^2)} t^{\alpha}\cdot\left(-\frac{1}{2}\right)\,\mathrm{d}t = \pi\left[-\frac{1}{\alpha+1}\,t^{\alpha+1}\right]_{a^2}^{a^2(1-p^2)} \\
&= \frac{\pi}{\alpha}\left\{a^{2(\alpha+1)} - a^{2(\alpha+1)}\cdot\left(1-p^2\right)^{\alpha+1}\right\} \\
&= \frac{\pi}{\alpha+1}\,a^{2(\alpha+1)}\left\{1-\left(1-p^2\right)^{\alpha+1}\right\}.
\end{aligned}$$

Since $\alpha + 1 > 0$ and $1 - p^2 \to 0+$ for $p \to 1-$, it follows that

$$\left(1-p^2\right)^{\alpha+1} \to 0 \qquad \text{for } p \to 1- .$$

Hence the integral is *convergent* for $\alpha > -1$ with the value

$$J(\alpha) = \int_B \left(a^2 - x^2 - y^2\right)^{\alpha} \, \mathrm{d}S = \frac{\pi}{\alpha+1}\, a^{2(\alpha+1)}.$$

Summing up we have proved that

$$J(\alpha) = \int_B \left(a^2 - x^2 - y^2\right)^{\alpha} \, \mathrm{d}S = \begin{cases} \dfrac{\pi}{\alpha+1}\, a^{2(\alpha+1)} & \text{for } \alpha > -1; \quad \text{convergence;} \\ \infty & \text{for } \alpha \le -1; \quad \text{divergence.} \end{cases}$$

When the integrand has fixed sign, we allow ourselves to put the value equal to $+\infty$ (positive integrand) or $-\infty$ (negative integrand).

Note that we shall *not* allow this notation, if both the positive part and the negative part are infinite, because $\infty - \infty$ does not make sense. We shall return to this in Example 30.5. $\Diamond$

Example 30.4

A. *In advanced technical literature one often sees the improper 1-dimensional integral*

$$\int_0^{\infty} e^{-x^2}\, dx.$$

Obvious applications can be found in Probability and Statistics (the normal distribution); but one can also find it in many other places (the heat equation, diffusion). We shall find the value of this important integral.

D. The integral cannot be calculated by methods from the elementary calculus. It is fairly easy to prove that it is convergent. In fact, if we introduce the function

$$f(t) = (1+t)e^{-t} \qquad \text{with } f'(t) = -t\,e^{-t} < 0 \quad \text{for } t > 0,$$

then it follows, because $f(t)$ is decreasing for $t > 0$, that we have

$$(1+t)\,e^{-t} \le f(0) = 1, \quad \text{i.e.} \quad e^{-t} \le \frac{1}{1+t} \text{ for } t \ge 0.$$

If we put $t = x^2$, we get

$$e^{-x^2} \le \frac{1}{1+x^2},$$

hence

$$0 < \int_0^n e^{-x^2}\,\mathrm{d}x \le \int_0^n \frac{\mathrm{d}x}{1+x^2} = \text{Arctan}\, n \to \frac{\pi}{2} \quad \text{for } n \to +\infty.$$

This proves the convergence, and also the estimate

$$(0 \le) \quad \int_0^{\infty} e^{-x^2}\,\mathrm{d}x \le \frac{\pi}{2}.$$

However, we still have not found the exact value.

We shall show that it is possible to find the value by using methods from the *Theory of Functions in Several Variables*. The trick is instead to consider the improper plane integral

$$I = \int_B \exp(-x^2-y^2)\,\mathrm{d}S, \qquad \text{where } B = [0,+\infty[^2 \text{ is the first quadrant.}$$

The integrand is defined and positive everywhere, so we shall not make any further division of the domain according to the sign of the integrand.

The domain of integration is unbounded, so we must truncate it in a bounded way. We have two obvious possibilities of doing this, depending on whether we consider *polar* or *rectangular* coordinates. The idea is to use both of them, because we by using the *polar* coordinates obtain the *value* of the integral I, and by using the *rectangular* coordinates we obtain the connection to the integral under consideration.

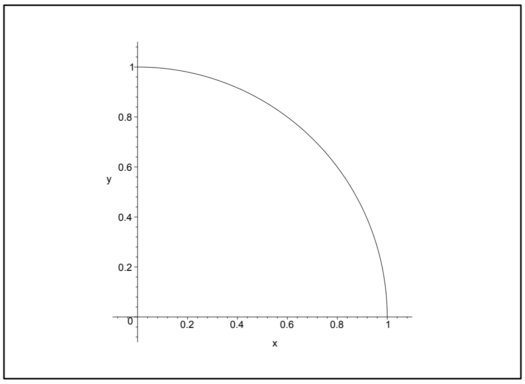

Figure 30.4: The domain $Q(R)$ for $R = 1$.

1) *Polar coordinates.* We truncate by taking the intersection of B and discs of radius R,

$$Q(R) = \{(x, y) \mid x \geq 0,\, y \geq 0,\, x^2 + y^2 \leq R^2\}.$$

We reach every point in the first quadrant B by taking the limit $R \to +\infty$.

When we apply the reduction theorem in polar coordinates over $Q(R)$ we get

$$\begin{aligned} I(R) &= \int_{Q(R)} \exp\left(-x^2 - y^2\right)\, \mathrm{d}S = \int_0^{\frac{\pi}{2}} \left\{ \int_0^R \exp\left(-\varrho^2\right)\, \varrho\, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= \frac{\pi}{2} \int_0^{R^2} e^{-t} \cdot \frac{1}{2}\, \mathrm{d}t = \frac{\pi}{4} \left[e^{-t}\right]_0^{R^2} = \frac{\pi}{4} \left\{1 - \exp\left(-R^2\right)\right\}. \end{aligned}$$

Since $\exp\left(-R^2\right) \to 0$ for $R \to +\infty$, we conclude that the improper integral is *convergent* with the value

$$I = \int_B \exp\left(-x^2 - y^2\right)\, \mathrm{d}S = \lim_{R \to \infty} I(R) = \frac{\pi}{4}.$$

2) *Rectangular coordinates.* The truncation is here

$$R(n) = \{(x, y) \mid 0 \leq x \leq n,\, 0 \leq y \leq n\} = [0, n]^2.$$

We get every point in the first quadrant B by taking the limit $n \to +\infty$.

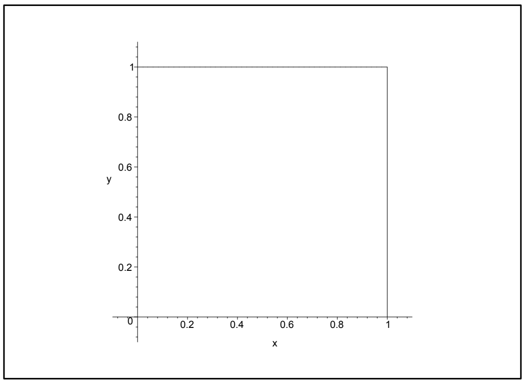

Figure 30.5: The domain $R(n)$ for $n = 1$.

When we apply the rectangular reduction theorem over $R(n)$ we get

$$\begin{aligned} J(n) &= \int_{R(n)} \exp\left(-x^2-y^2\right)\,\mathrm{d}S = \int_0^n \left\{ \int_0^n e^{-x^2}\cdot e^{-y^2}\,\mathrm{d}x \right\}\mathrm{d}y \\ &= \int_0^n e^{-x^2}\,\mathrm{d}x \cdot \int_0^n e^{-y^2}\,\mathrm{d}y = \left\{ \int_0^n e^{-t^2}\,\mathrm{d}t \right\}^2 . \end{aligned}$$

3) *Summary.* According to 1) the improper integral is convergent, hence the limit can be taken in 2). Since the limit is the same, no matter which truncation we are using, we must have

$$I = \frac{\pi}{4} = \int_B \exp\left(-x^2-y^2\right)\,\mathrm{d}S = \lim_{n\to\infty} J(n) = \left\{ \int_0^\infty e^{-t^2}\,\mathrm{d}t \right\}^2 .$$

Since $\int_0^\infty e^{-t^2}\,\mathrm{d}t > 0$, we finally get the value of the integral

$$\int_0^\infty e^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}.$$

Remark 30.4 It follows that

$$I = \frac{\sqrt{\pi}}{2} \le \frac{\pi}{2},$$

cf. a remark in section **D**. $\Diamond$

Example 30.5

A. *Examine whether the improper plane integral*

$$\int_{\mathbb{R}^2} \frac{xy}{(1+x^2)(1+y^2)}\,\mathrm{d}S$$

is convergent or divergent.

D. An analysis of the sign shows that the integrand is positive in the first and the third quadrant, while it is negative in the second and the fourth quadrant, so we shall also divide the domain according to the sign. Here we shall also demonstrate the *wrong* argument where one forgets to divide according to the sign of the integrand. This shows that one has to be careful here.

I. We see that the integrand changes its sign if we reflect it in either the x-axis or in the y-axis. This shows that the integral over *any* bounded set B, which is symmetric with respect to at least one of the axes, is 0,

$$\int_B \frac{xy}{(1+x^2)(1+y^2)}\,\mathrm{d}S = 0.$$

The usual truncations in discs or centred squares satisfy this symmetry with respect to both axes, so if one is not too careful one will *erroneously* conclude in the limit that the plane integral is "convergent with the value $\lim_{n\to\infty} 0 = 0$".

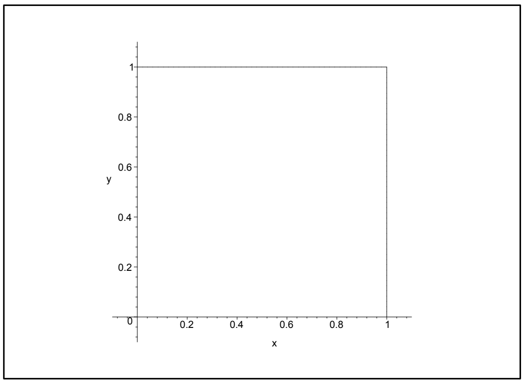

Figure 30.6: The truncation $B(n)$ for $n = 1$.

We shall prove that this conclusion is *not* correct. We choose this time the truncation

$$B(n) = [0, n] \times [0, n],$$

i.e. we only consider a subset of the set, where the integrand is ≥ 0. If something goes wrong here in the first quadrant, then it also is wrong in any bigger subset of $[0, n]^2 \cup [-n, 0]^2$.

By reduction over $B(n)$ we get

$$\begin{aligned}
\int_{B(n)} \frac{xy}{(1+x^2)(1+y^2)}\,\mathrm{d}S &= \int_0^n \left\{ \int_0^n \frac{xy}{(1+x^2)(1+y^2)}\,\mathrm{d}x \right\} \mathrm{d}y \\
&= \int_0^n \frac{x}{1+x^2}\,\mathrm{d}x \cdot \int_0^n \frac{y}{1+y^2}\,\mathrm{d}y = \left\{ \int_0^n \frac{t}{1+t^2}\,\mathrm{d}t \right\}^2 \\
&= \left\{ \left[\frac{1}{2} \ln\left(1+x^2\right) \right]_0^n \right\}^2 = \frac{1}{4} \left\{ \ln\left(1+n^2\right) \right\}^2 \\
&\to +\infty \qquad \text{for } n \to \infty.
\end{aligned}$$

This shows that the improper integral is *divergent*, thus our first argument must be wrong! (Of the type $\infty - \infty$).

Remark 30.5 We shall demonstrate how wrong this illegal method is. If we choose the "skew" truncation

$$Q(a, n) = [-a\,n, n] \times [-an, n], \qquad a > 0 \text{ constant},$$

we still get $\mathbb{R}^2$ by taking the limit $n \to +\infty$. We get by a rectangular reduction,

$$\begin{aligned}
\int_{Q(a,n)} \frac{xy}{(1+x^2)(1+y^2)}\,\mathrm{d}S &= \left\{ \int_{-an}^n \frac{t}{1+t^2}\,\mathrm{d}t \right\}^2 = \left\{ \left[\frac{1}{2} \ln\left(1+t^2\right) \right]_{-an}^n \right\}^2 \\
&= \left\{ \frac{1}{2} \ln\left(\frac{1+n^2}{1+a^2n^2} \right) \right\}^2 = \left\{ \frac{1}{2} \ln\left(\frac{1+\frac{1}{n^2}}{a^2+\frac{1}{n^2}} \right) \right\}^2 \\
&\to \{\ln a\}^2 \qquad \text{for } n \to \infty.
\end{aligned}$$

Every calculation is correct. The only thing which is *wrong* is that the assumptions of taking the limit (with respect to the conclusion of convergence) are *not* satisfied. We note that $\{\ln a\}^2$ go through the whole interval $[0, +\infty[$, when a go through $\mathbb{R}_+$, which means that we can obtain any $q \geq 0$ as a candidate for a limit of the improper plane integral, which is nonsense.

If we instead use the truncations

$$R(a,n) = [-n, an] \times [-an, n], \qquad a > 0 \text{ constant},$$

we obtain analogously all negative numbers as possible limits. But *if* the limit exists, then it is unique! Hence the improper plane integral is *divergent*. $\Diamond$

Example 30.6 *Given the meridian curve $\mathcal{M}$ of the parametric description*

$$\varrho = a\,\cos t, \quad z = a\{\ln(1+\sin t) - \ln\cos t - \sin t\}, \qquad t \in \left[0, \frac{\pi}{2}\right[$$

When this is rotated we obtain a surface of revolution $\mathcal{O}$ (half of the pseudo-sphere), which stretches into infinity along the positive part of the Z-axis.
Find the integral which gives the area of that part of $\mathcal{O}$, which corresponds to $[0, T]$, where $T < \frac{\pi}{2}$.
Then find the area of the pseudo-sphere by letting $T \to \frac{\pi}{2}$.

A Surface area of an infinite surface of revolution; improper surface integral.

D First find the curve element $\mathrm{d}s$ on $\mathcal{M}$. Then compute the surface area of $\mathcal{O}_T$, i.e. the surface corresponding to $t \in [0, T]$, where $T < \frac{\pi}{2}$. This means that we shall calculate

$$2\pi \int_{\mathcal{O}_T} \varrho(t)\,ds.$$

Finally, take the limit $T \to \frac{\pi}{2}-$.

I First calculate

$$\begin{aligned} \mathbf{r}'(t) &= a\left(-\sin t, \frac{\cos t}{1+\sin t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(-\sin t, \frac{\cos t\cdot(1-\sin t)}{1-\sin^2 t} + \frac{\sin t}{\cos t} - \cos t\right) \\ &= a\left(-\sin t, \frac{1-\sin t+\sin t-\cos^2 t}{\cos t}\right) = a\,\sin t\cdot(-1, \tan t). \end{aligned}$$

Hence

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = a|\sin t|\sqrt{1+\tan^2 t}\,\mathrm{d}t = a\left|\frac{\sin t}{\cos t}\right|\,\mathrm{d}t = a\,\tan t\,\mathrm{d}t,$$

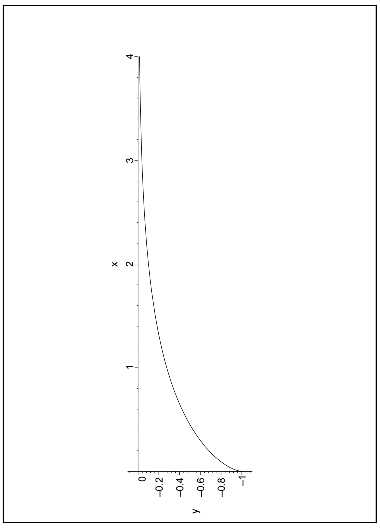

Figure 30.7: The meridian curve of the pseudo-sphere.

and accordingly,

$$\begin{aligned}\text{area}\,(\mathcal{O}_T) &= 2\pi \int_{\mathcal{O}_T} \varrho(t)\,\mathrm{d}s = 2\pi \int_0^T a\,\cos t \cdot a\,\tan t\,\mathrm{d}t \\ &= 2\pi a^2 \int_0^T \sin t\,dt = 2\pi a^2(1-\cos T).\end{aligned}$$

Finally, by taking the limit we find the improper surface area

$$\text{area}(\mathcal{O}) = \lim_{T\to\frac{\pi}{2}-} \text{area}\,(\mathcal{O}_T) = 2\pi a^2.$$

Remark. Note that the "half" pseudo-sphere" has the same surface area as the usual upper half sphere of radius a. $\Diamond$

Example 30.7 *Check in each of the following cases if the given surface integral is convergent or divergent; in case of convergency, find the value.*

1) *The surface integral* $\int_{\mathcal{F}} \frac{1}{(a+4z)^2}\,\mathrm{d}S$ *over the surface* $\mathcal{F}$ *given by* $az = x^2+y^2$, $(x,y)\in\mathbb{R}^2$.

2) *The surface integral* $\int_{\mathcal{F}} \frac{x^2}{z^2+a^2}\,\mathrm{d}S$ *over the surface* $\mathcal{F}$ *given by* $x^2+y^2=a^2$, $z\in\mathbb{R}$.

3) *The surface integral* $\int_{\mathcal{F}} y^2 \exp\left(-\frac{|z|}{a}\right)\mathrm{d}S$ *over the surface* $\mathcal{F}$ *given by* $x^2+y^2=a^2$, $z\in\mathbb{R}$.

4) *The surface integral* $\int_{\mathcal{F}} \frac{1}{z(x+y)}\,\mathrm{d}S$ *over the surface* $\mathcal{F}$ *given by* $z=\sqrt{2xy}$, $(x,y)\in[a,+\infty[^2$.

A Improper surface integral.

D First analyze why the integral is improper. Then truncate the surface and split it into the positive and the negative part of the integrand. Finally take the limit.

I 1) The surface is a paraboloid of revolution.

$$z = \frac{1}{a}\,(x^2+y^2) = \frac{1}{a}\,\varrho \geq 0.$$

The integrand is $\geq \frac{1}{a^2} > 0$ everywhere on the surface.

The surface is described as the graph of the equation $z = \frac{1}{a}\,(x^2+y^2)$, so the weight function becomes

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\left(\frac{2x}{a}\right)^2+\left(\frac{2y}{a}\right)^2} = \sqrt{1+\frac{4}{a^2}\,\varrho^2}.$$

We choose the truncated domain in polar coordinates as $0\leq\varrho\leq R$. It follows from the above that the area element is

$$\mathrm{d}S = \sqrt{1+\frac{4}{a^2}\,\varrho^2}\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi,$$

hence the surface integral over the truncated surface $\mathcal{F}_R$ is

$$\begin{aligned}
\int_{\mathcal{F}_R} \frac{1}{(a+4z)^2}\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^R \frac{1}{\left(a+\frac{4\varrho^2}{a}\right)^2}\sqrt{1+\frac{4}{a^2}\,\varrho\,\mathrm{d}\varrho}\right\}\mathrm{d}\varphi \\
&= \frac{2\pi}{a^2}\int_0^R\left\{1+\frac{4}{a^2}\,\varrho^2\right\}^{-\frac{3}{2}}\cdot\frac{1}{2}\cdot\frac{a^2}{4}\cdot\frac{4}{a^2}\cdot 2\varrho\,\mathrm{d}\varrho \\
&= \frac{2\pi}{a^2}\cdot\frac{a^2}{8}\left[-\frac{2}{\sqrt{1+\frac{4}{a^2}\,\varrho^2}}\right]_0^R = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+\frac{4}{a^2}\,R^2}}\right).
\end{aligned}$$

This expression clearly converges for $R\to+\infty$, hence the improper surface integral is convergent of the value

$$\int_{\mathcal{F}} \frac{1}{(a+4z)^2}\,\mathrm{d}S = \lim_{R\to+\infty}\frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+\frac{4}{a^2}\,R^2}}\right) = \frac{\pi}{2}.$$

2) The surface is an infinite cylinder surface with the circle in the XY-plane of centrum $(0,0)$ and radius a as its leading curve. When we use semi-polar coordinates we get

$$x = a\,\cos\varphi,\quad y = a\,\sin\varphi,\quad z=z,\qquad \varphi\in[0,2\pi],\quad z\in\mathbb{R},$$

and

$$\mathrm{d}S = a\,\mathrm{d}\varphi\,\mathrm{d}z.$$

The integrand is positive, so we choose the truncation $|z|\le A$. Then

$$\begin{aligned}
\int_{\mathcal{F}_A}\frac{x^2}{z^2+a^2}\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_{black-A}^{A}\frac{a^2\cos^2\varphi}{z^2+a^2}\cdot a\,\mathrm{d}z\right\}\mathrm{d}\varphi \\
&= \int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\cdot a^2\int_{-A}^{A}\frac{1}{1+\left(\frac{z}{a}\right)^2}\cdot\frac{1}{a}\,\mathrm{d}z = a^2\pi\cdot 2\mathrm{Arctan}\left(\frac{A}{a}\right).
\end{aligned}$$

This expression converges for $A\to+\infty$, and we conclude that the improper surface integral is convergent with the value

$$\int_{\mathcal{F}}\frac{x^2}{z^2+a^2}\,\mathrm{d}S = \lim_{A\to+\infty}a^2\pi\cdot 2\mathrm{Arctan}\left(\frac{A}{a}\right) = a^2\pi^2.$$

3) By using semi-polar coordinates it is seen that

$$x = a\,\cos\varphi,\quad y = a\,\sin\varphi,\quad z=z,\qquad \varphi\in[0,2\pi],\quad z\in\mathbb{R}.$$

The surface element is

$$\mathrm{d}S = a\,\mathrm{d}\varphi\,\mathrm{d}z.$$

The integrand is positive everywhere, so we choose the truncation $|z| \le A$. Then

$$\begin{aligned}
\int_{\mathcal{F}_A} y^2 \exp\left(-\frac{|z|}{a}\right) \mathrm{d}S &= \int_0^{2\pi} \left\{ \int_{-A}^{A} a^2 \sin^2\varphi \cdot \exp\left(-\frac{|z|}{a}\right) a\, \mathrm{d}z \right\} \mathrm{d}\varphi \\
&= a^3 \cdot \pi \cdot 2 \int_0^A \exp\left(-\frac{z}{a}\right) \mathrm{d}z = 2a^4\pi \cdot \left[-\exp\left(-\frac{z}{a}\right)\right]_0^A \\
&= 2\pi a^4 \left\{ 1 - \exp\left(-\frac{A}{a}\right)\right\}.
\end{aligned}$$

This expression is clearly convergent for $A \to +\infty$, hence the improper surface integral is convergent with the value

$$\int_{\mathcal{F}} y^2 \exp\left(-\frac{|z|}{a}\right) \mathrm{d}S = \lim_{A\to+\infty} \int_{\mathcal{F}_A} y^2 \exp\left(-\frac{|z|}{a}\right) \mathrm{d}S = 2\pi a^4.$$

4) When the surface is the graph of $z = \sqrt{2xy}$ for $x, y \ge a$, then the surface element is

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}x\, \mathrm{d}y = \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\, \mathrm{d}x\, \mathrm{d}y \\
&= \sqrt{\frac{2xy + y^2 + x^2}{2xy}}\, \mathrm{d}x\, \mathrm{d}y = \frac{x+y}{\sqrt{2xy}}\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

The integrand is given on the surface $\mathcal{F}$ by

$$\frac{1}{z(x+y)} = \frac{1}{\sqrt{2xy}(x+y)}$$

which is clearly positive, because $x,\, y \ge a$.
For every $A > a$ we define the truncation by $a \le x,\, y \le A$. Then the surface integral over the corresponding truncated surface $\mathcal{F}_A$ is

$$\int_{\mathcal{F}_A} \frac{1}{z(x+y)}\,\mathrm{d}S = \int_a^A \left\{ \int_a^A \frac{1}{\sqrt{2xy}(x+y)} \cdot \frac{x+y}{\sqrt{2xy}}\,\mathrm{d}x \right\} \mathrm{d}y$$
$$= \frac{1}{2}\left\{\int_a^A \frac{\mathrm{d}x}{x}\right\} \cdot \left\{\int_a^A \frac{\mathrm{d}y}{y}\right\} = \frac{1}{2}\left\{[\ln t]_a^A\right\}^2 = \frac{1}{2}\left\{\ln\left(\frac{A}{a}\right)\right\}^2 \to +\infty \quad \text{for } A \to +\infty.$$

We conclude that the improper surface integral is divergent.

5) The surface is the same as in **Example 30.7.4**, so the surface element is

$$\mathrm{d}S = \frac{x+y}{\sqrt{2xy}}\,\mathrm{d}x\,\mathrm{d}y.$$

The integrand is on the surface $\mathcal{F}$ given by

$$\frac{1}{z^2xy} = \frac{1}{2xy \cdot xy} = \frac{1}{2} \cdot \frac{1}{x^2} \cdot \frac{1}{y^2}.$$

This is positive, so we shall again use the truncation $a \le x,\, y \le A$. Then

$$\int_{\mathcal{F}_A} \frac{1}{z^2xy}\,\mathrm{d}S = \int_a^A \left\{ \int_a^A \frac{1}{2} \cdot \frac{1}{x^2} \cdot dfrac1y^2 \cdot \frac{x+y}{\sqrt{2xy}}\,dx \right\} \mathrm{d}y$$
$$= \frac{1}{2\sqrt{2}} \int_a^A x^{-\frac{3}{2}}\,\mathrm{d}x \cdot \int_a^A y^{-\frac{5}{2}}\,\mathrm{d}y + \frac{1}{2\sqrt{2}} \int_a^A x^{-\frac{5}{2}}\,\mathrm{d}x \cdot \int_a^A y^{-\frac{3}{2}}\,\mathrm{d}y$$
$$= \frac{1}{\sqrt{2}}\left[-\frac{2}{\sqrt{x}}\right]_a^A \cdot \left[-\frac{2}{3} \cdot \frac{1}{y\sqrt{y}}\right]_a^A = \frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{A}}\right) \cdot \left(\frac{1}{a\sqrt{a}} - \frac{1}{A\sqrt{A}}\right)$$
$$\to \frac{2\sqrt{2}}{3} \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{a\sqrt{a}} = \frac{2\sqrt{2}}{3a^2} \qquad \text{for } A \to +\infty.$$

The improper surface integral converges towards the value

$$\int_{\mathcal{F}} \frac{1}{z^2xy}\,\mathrm{d}S = \frac{2\sqrt{2}}{3a^2}.$$

Example 30.8 *Check in each of the following cases if the given surface integral is convergent or divergent; in case of convergency, find its value.*
Let $\mathcal{S}$ denote the sphere of centrum $(0,0,0)$ and radius a, while $\mathcal{F}$ is given by $az = x^2+y^2$, $x^2+y^2 \le a^2$.

1) $\int_{\mathcal{S}} \frac{1}{a-z}\,\mathrm{d}S$,

2) $\int_{\mathcal{S}} \sqrt{\frac{a}{|z|}}\,\mathrm{d}S$,

3) $\int_{\mathcal{F}} \frac{1}{a-z}\,\mathrm{d}S$,

4) $\int_{\mathcal{F}} \sqrt{\frac{a}{z}}\,\mathrm{d}S$.

A Improper surface integrals.

D Analyze why the integral is improper. Since the integrands are ≥ 0 in all cases, we shall only find some nice truncations of the surface.

I 1) Since $|z| \le a$ on $\mathcal{S}$, the integrand is $\frac{1}{a-z} > 0$ on $\mathcal{S} \setminus \{(0,0,a)\}$. The integrand tends towards $+\infty$, when $(x,y,z) \to (0,0,a)$ on $\mathcal{S}$.
When we use spherical coordinates on $\mathcal{S}$,

$$x = a\,\cos\varphi \cdot \sin\theta, \quad y = a\,\sin\varphi \cdot \sin\theta, \quad z = a\,\cos\theta,$$

for

$$\varphi \in [0, 2\pi], \qquad \theta \in [0, \pi],$$

it is well-known that

$$\mathrm{d}S = a^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

The singular point $(0,0,a)$ corresponds to $\theta = 0$, hence we choose the truncation $\theta \in [\varepsilon, \pi]$, where $\varepsilon > 0$ corresponds to the subsurface $\mathcal{S}_\varepsilon$. When we integrate over $\mathcal{S}_\varepsilon$ we get

$$\begin{aligned}
\int_{\mathcal{S}_\varepsilon} \frac{1}{a-z}\,\mathrm{d}S &= \int_0^{2\pi} \left\{ \int_\varepsilon^\pi \frac{1}{a - a\,\cos\theta}\, a^2 \sin\theta\,\mathrm{d}\theta \right\} \mathrm{d}\varphi \\
&= 2\pi a \int_\varepsilon^\pi \frac{\sin\theta}{1-\cos\theta}\,\mathrm{d}\theta = 2\pi a\,[\ln(1-\cos\theta)]_\varepsilon^\pi \\
&= 2\pi a\{\ln 2 - \ln(1-\cos\varepsilon)\} = 2\pi a \ln \frac{2}{2\sin^2 \frac{\varepsilon}{2}} \\
&= 4a\pi \ln \frac{1}{\sin \frac{\varepsilon}{2}} \to +\infty \qquad \text{for } \varepsilon \to 0+,
\end{aligned}$$

and the improper surface integral is divergent.

2) In this case the integrand is > 0 on $\mathcal{S}_0$, where $\mathcal{S}_0$ is the set of points on $\mathcal{S}$, which is not contained in the XY-plane, where the integrand is not defined. We use again spherical coordinates. Due to the symmetry it suffices to consider the domain

$$\mathcal{S}_\varepsilon : \qquad \varphi \in [0, 2\pi] \quad \text{and} \quad \theta \in \left[0, \frac{\pi}{2} - \varepsilon\right].$$

It follows by insertion of $z = a\,\cos\theta$ and $\mathrm{d}S = a^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, that

$$\begin{aligned}\int_{\mathcal{S}_\varepsilon} \sqrt{\frac{a}{|z|}}\,\mathrm{d}S &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}-\varepsilon} \sqrt{\frac{a}{a\,\cos\theta}} \cdot a^2 \sin\theta\,\mathrm{d}\theta \right\} \mathrm{d}\varphi \\ &= 2\pi a^2 \int_0^{\frac{\pi}{2}-\varepsilon} \frac{\sin\theta}{\sqrt{\cos\theta}}\,\mathrm{d}\theta = 2\pi a^2 \left[-2\sqrt{\cos\theta} \right]_0^{\frac{\pi}{2}-\varepsilon} \\ &= 4\pi a^2 \{1 - \sqrt{\sin\varepsilon}\} \to 4\pi a^2 \qquad \text{for } \varepsilon \to 0+\,.\end{aligned}$$

We conclude that the improper surface integral is convergent.
Of symmetric reasons the value is

$$\int_{\mathcal{S}} \sqrt{\frac{a}{|z|}}\,\mathrm{d}S = \lim_{\varepsilon\to 0+} 2\int_{\mathcal{S}_\varepsilon} \sqrt{\frac{a}{|z|}}\,\mathrm{d}S = 2\cdot 4\pi a^2 = 8\pi a^2.$$

3) The surface is the graph of $z = \frac{1}{a}\left(x^2+y^2\right) = \frac{\varrho^2}{a}$, so the area element is

$$\begin{aligned}\mathrm{d}S &= \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1+\left(\frac{2x}{a}\right)^2+\left(\frac{2y}{a}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{a}\sqrt{a^2+4(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{a}\sqrt{a^2+4\varrho^2}\cdot\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi.\end{aligned}$$

The integrand is the same as in **Example 30.8.1**, and since $z \le a$ on $\mathcal{F}$, it is positive for $z < a$. We choose the truncation in polar coordinates by

$$\mathcal{F}_\varepsilon: \qquad 0 \le \varrho \le a-\varepsilon, \quad \varphi \in [0, 2\pi].$$

Then by insertion,

$$\begin{aligned}\int_{\mathcal{F}_\varepsilon} \frac{1}{a-z}\,\mathrm{d}S &= \int_0^{2\pi} \left\{ \int_0^{a-\varepsilon} \frac{1}{a-\frac{\varrho^2}{a}} \cdot \frac{1}{a}\sqrt{a^2+4\varrho^2}\,\varrho\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\ &= 2\pi \int_0^{a-\varepsilon} \frac{\varrho}{a^2-\varrho^2}\sqrt{a^2+4\varrho^2}\,\mathrm{d}\varrho \ge a\pi \int_0^{a-\varepsilon} \frac{1}{a^2-\varrho^2}\cdot 2\varrho\,\mathrm{d}\varrho \\ &= a\pi \left[-\ln\left(a^2-\varrho^2\right) \right]_0^{a-\varepsilon} \\ &= a\pi \left\{ \ln a^2 - \ln\left(a^2-(a-\varepsilon)^2\right) \right\} \to +\infty\end{aligned}$$

for $\varepsilon \to 0+$, and the improper surface integral is divergent.

4) The singular point is $(0,0,0)$. We choose the truncation

$$\mathcal{F}_\varepsilon: \qquad \varphi \in [0, 2\pi], \quad \varrho \in [\varepsilon, a],$$

and

$$z = \frac{\varrho^2}{a} > 0, \qquad \mathrm{d}S = \frac{1}{a}\sqrt{a^2+4\varrho^2}\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi.$$

Then by insertion

$$
\begin{aligned}
\int_{\mathcal{F}_\varepsilon} \sqrt{\frac{a}{z}}\, \mathrm{d}S &= \int_0^{2\pi} \left\{ \int_\varepsilon^a \sqrt{\frac{a^2}{\varrho^2}} \cdot \frac{1}{a} \sqrt{a^2+4\varrho^2}\, \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= 2\pi a \int_\varepsilon^a \sqrt{1+\left(\frac{2\varrho}{a}\right)^2}\, \mathrm{d}\varrho \qquad \left[\frac{2\varrho}{a} = \sinh t\right] \\
&= 2\pi a \int_{\varrho=\varepsilon}^a \sqrt{1+\sinh^2 t} \cdot \frac{a}{2} \cosh t \, \mathrm{d}t = \pi a^2 \int_{\mathrm{Arsinh}\, \frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\, 2} \cosh^2 t \, \mathrm{d}t \\
&= \frac{\pi a^2}{2} \int_{\mathrm{Arsinh}\, \frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\, 2} (1+\cosh 2t)\, \mathrm{d}t = \frac{\pi a^2}{2} \left[t + \frac{1}{2} \sinh 2t \right]_{\mathrm{Arsinh}\, \frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\, 2} \\
&= \frac{\pi a^2}{2} \left\{ \mathrm{Arsinh}\, 2 - \mathrm{Arsinh}\, \frac{2\varepsilon}{a} \right\} + \frac{\pi a^2}{2} \left[\sinh t \sqrt{1+\sinh^2 t} \right]_{\mathrm{Arsinh}\, \frac{2\varepsilon}{a}}^{\mathrm{Arsinh}\, 2} \\
&\to \frac{\pi a^2}{2}\, \mathrm{Arsinh}\, 2 + \frac{\pi a^2}{2} \{ \ln(2+\sqrt{5}) + 2\sqrt{5} \}
\end{aligned}
$$

for $\varepsilon \to 0+$, and the improper surface integral converges towards the value

$$\int_{\mathcal{F}} \sqrt{\frac{a}{z}}\, \mathrm{d}S = \frac{\pi a^2}{2} \{ \ln(2+\sqrt{5}) + 2\sqrt{5} \}.$$

Example 30.9 *Check if the surfaces of the bodies of revolution of the leading curves of the equations*

$$y^2(a-x)=x^3 \quad (\textit{rectangular}) \qquad \textit{and} \qquad \varrho=\frac{a^2}{a^2+z^2}, \quad z\in\mathbb{R}, \quad (\textit{Semi-polar coordinates}),$$

can be given a finite area.
(The values shall not be computed).

A Improper surface integrals.

D Since we are only dealing with areas, the integrand is automatically positive. Truncate suitably before the computation of the surface integral, and then take the limit.

I 1) The curve $\mathcal{K}$ of the equation

$$y^2(a-x)=x^3$$

is rotated around the asymptote $x=a$.
For symmetric reasons it suffices to consider $y\geq 0$, thus

$$y=x\sqrt{\frac{x}{a-x}}=x^{\frac{3}{2}}(a-x)^{-\frac{1}{2}}.$$

One easily sees that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\sqrt{\frac{x}{(a-x)^3}}\cdot(3a-2x).$$

The length of the circle C_x (around the line $x=a$) at the height $y(x)$ is $2\pi(a-x)$, [In fact, $0\leq x<a$].
If we truncate at the height $y(x_0)$ corresponding to some $x_0\in[0,a[$, and remember the symmetry around $y=0$ we get the corresponding surface area,

$$\begin{aligned}
&2\int_0^{x_0}\text{length}\,(C_x)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x\\
&\quad=2\int_0^{x^0}2\pi(a-x)\cdot\frac{1}{2}\sqrt{\frac{x}{(a-x)^3}}\cdot(3a-2x)\,\mathrm{d}x=2\pi\int_0^{x_0}\sqrt{\frac{x}{a-x}}\,\{a+2(a-x)\}\,\mathrm{d}x\\
&\quad=2\pi\int_0^{x_0}\left\{a\sqrt{\frac{x}{a-x}}+2\sqrt{x(a-x)}\right\}\mathrm{d}x, \qquad 0<x_0<a.
\end{aligned}$$

We conclude that the surface has a finite area. The only problem is the term $\sqrt{\dfrac{x}{a-x}}$ in the integrand, and

$$0\leq\sqrt{\frac{x}{a-x}}\leq\sqrt{a}\cdot\frac{1}{\sqrt{a-x}} \qquad \text{for } 0<x<a,$$

and

$$\int_0^{x_0}\frac{dx}{\sqrt{a-x}}=\left[-2\sqrt{a-x}\right]_0^{x_0}=2\{\sqrt{a}-\sqrt{a-x_0}\},$$

which converges towards $2\sqrt{a}$ for $x_0\to a$. Since the area of the surface is smaller than this value, we conclude that the improper surface integral exists.

Remark. It is in fact possibl to find the exact value. When we put $t = \dfrac{x}{a-x}$ we get

$$x = \frac{at}{t+t} = a - \frac{a}{t+1},$$

hence

$$\mathrm{d}x = \frac{a}{(t+1)^2}\,\mathrm{d}t,$$

and

$$\begin{aligned}2\pi \int a\sqrt{\frac{x}{a-x}}\,\mathrm{d}x &= 2\pi a \int \sqrt{t}\cdot\frac{a}{(t+1)^2}\,\mathrm{d}t = 2\pi a^2 \int u\cdot\frac{1}{(u^2+1)^2}\cdot 2u\,\mathrm{d}u \\ &= 2\pi a^2\left\{-\frac{u}{u^2+1}+\int\frac{1}{u^2+1}\,\mathrm{d}u\right\} = 2\pi a^2\left\{\operatorname{Arctan} u - \frac{u}{u^2+1}\right\} \\ &= 2\pi a^2\left\{\operatorname{Arctan}\sqrt{\frac{x}{a-x}} - \frac{\sqrt{\dfrac{x}{a-x}}}{\dfrac{x}{a-x}+1}\right\} = 2\pi a^2\left\{\operatorname{Arctan}\sqrt{\frac{x}{a-x}} - \frac{1}{a}\sqrt{x(a-x)}\right\},\end{aligned}$$

hence by taking the limit

$$2\pi\int_0^a a\sqrt{\frac{x}{a-x}}\,\mathrm{d}x = 2\pi a^2\left\{\frac{\pi}{2}-0\right\} = \pi^2 a^2.$$

The latter integral is calculated by noting that $y = \sqrt{x(a-x)}$ for $0 \le x \le a$ describes a half circle of centrum $\dfrac{a}{2}$ and radius $\dfrac{a}{2}$, hence

$$4\pi\int_0^a \sqrt{x(a-x)}\,\mathrm{d}x = 4\pi\cdot\frac{1}{2}\cdot\pi\left(\frac{a}{2}\right)^2 = \frac{\pi^2a^2}{2}.$$

Summarizing, the improper surface area is convergent, and its value is

$$\pi^2a^2 + \frac{\pi^2a^2}{2} = \frac{3}{2}\,\pi^2a^2. \qquad \Diamond$$

2) When the curve $\varrho = \dfrac{a^2}{a^2+z^2}$, $z \in \mathbb{R}$, is rotated around the Z-axis, we get an infinite surface which at the height z is cut into a circle $C(x)$ of radius $\varrho(z)$, hence

$$\operatorname{length}(C(x)) = 2\pi\varrho = \frac{2\pi a^3}{a^2+z^2}.$$

When we put

$$\mathcal{F}_k = \{(x,y,z)\in\mathcal{F} \mid |z| \le ka\}, \qquad k > 0,$$

then we get

$$\begin{aligned}\operatorname{area}(\mathcal{F}_k) &= 2\int_0^{ka}\operatorname{length}(C(x))\,\mathrm{d}z = 4\pi a^3\int_0^{ka}\frac{1}{a^2+z^2}\,\mathrm{d}z \\ &= 4\pi a^2\int_0^k\frac{1}{1+t^2}\,\mathrm{d}t = 4\pi a^2\operatorname{Arctan} k \\ &\to 4\pi a^2\cdot\frac{\pi}{2} = 2\pi^2a^2 \quad \text{for } k\to+\infty.\end{aligned}$$

The improper surface area exists and its value is

$$\text{area}(\mathcal{F}) = 2\pi^2 a^2.$$

Example 30.10 *A surface $\mathcal{F}$ is given by the equation*

$$z = 1 + x^2 - y^2, \qquad (x, y) \in \mathbb{R}^2.$$

1. *Indicate the type of the surface and its vertices.*

2. *Find an equation of the tangent plane of $\mathcal{F}$ through the point $(2, 1, 4)$.*

Let q be a positive number. Let $\mathcal{F}(q)$ denote the subset of $\mathcal{F}$, which is given by

$$z = 1 + x^2 - y^2, \qquad x^2 + y^2 \le q^2.$$

3. *Compute the surface integral*

$$I(q) = \int_{\mathcal{F}(q)} \frac{1}{(z + 3x^2 + 5y^2)^{3/2}}\,\mathrm{d}S.$$

4. *Explain shortly why*

$$I = \int_{\mathcal{F}} \frac{1}{(z + 3x^2 + 5y^2)^{3/2}}\,\mathrm{d}S$$

is an improper surface integral and prove that I is divergent.

A Surface; tangent plane; surface integral; improper surface integral.

D Identify the type of the surface; e.g. set up a parametric description (or use a formula) and find find the field of the normal vectors. Calculate the surface integral by a reduction theorem. Note that the integrand is positive, and finally take the limit.

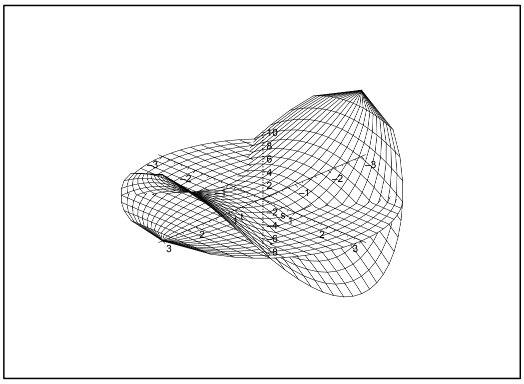

Figure 30.8: The surface $\mathcal{F}(q)$ for $q = 3$ with the projection $D(q)$ onto the (x, y)-plane.

I 1) It follows from the rearrangement

$$z - 1 = x^2 - y^2$$

that the surface is an equilateral hyperbolic paraboloid with its vertex at $(0, 0, 1)$.

2) It follows from the parametric description

$$\mathbf{r}(x, y) = \left(x, y, 1 + x^2 - y^2\right), \qquad (x, y) \in \mathbb{R}^2,$$

that

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = (-2x, 2y, 1).$$

Then we check if the point $(2, 1, 4)$ lies on $\mathcal{F}$:

$$1 + x^2 - y^2 = 1 + 4 - 1 = 4 = z,$$

thus $(2, 1, 4) \in \mathcal{F}$.

The normal vector is in this point

$$(-2x, 2y, 1) = (-4, 2, 1) = \mathbf{N},$$

and an equation of the tangent plane is

$$\begin{aligned} 0 &= \mathbf{N} \cdot (x-2, y-1, z-4) = (-4, 2, 1) \cdot (x-2, y-1, z-4) \\ &= -4x + 2y + z + 8 - 2 - 4 = -4x + 2y + z + 2, \end{aligned}$$

hence by a rearrangement,

$$z = 4x - 2y - 2.$$

3) The parametric domain for $\mathcal{F}(q)$ is the disc in the (x, y)-plane

$$D(q) = \{(x, y) \mid x^2 + y^2 \leq q^2\}.$$

Since $z = 1 + x^2 - y^2$ on $\mathcal{F}(q)$, it follows by the theorem of reduction that

$$\begin{aligned} I(q) &= \int_{\mathcal{F}(q)} \frac{1}{(z+3x^2+5y^2)^{3/2}} \, \mathrm{d}S = \int_{D(q)} \frac{\|\mathbf{N}(x,y)\|}{(1+4x^2+4y^2)^{3/2}} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{D(q)} \frac{(1+4x^2+4y^2)^{1/2}}{(1+4x^2+4y^2)^{3/2}} \, \mathrm{d}x \, \mathrm{d}y = \int_{D(q)} \frac{1}{1+4(x^2+y^2)} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^{2\pi} \left\{ \int_0^q \frac{1}{1+4\varrho^2} \, \varrho \, \mathrm{d}\varrho \right\} \mathrm{d}\varphi = 2\pi \cdot \frac{1}{8} \ln\left(1 + 4q^2\right) = \frac{\pi}{4} \ln\left(1 + 4q^2\right). \end{aligned}$$

4) Now $\mathcal{F}$ is unbounded, so I is an improper surface integral. The integrand is positive on $\mathcal{F}$, hence it suffices to take the limit $q \to +\infty$ for $I(q)$. Then

$$I = \lim_{q \to +\infty} I(q) = \frac{\pi}{4} \lim_{q \to +\infty} \ln\left(1 + 4q^2\right) = +\infty,$$

which proves that the improper surface integral is divergent.

31 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

31.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}$$

31.2 Powers etc.

Logarithm:

$$\begin{array}{rcll}
\ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| &=& r\ln|x|, & x\neq 0.
\end{array}$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \qquad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$\begin{array}{ll}
a^x\cdot a^y = a^{x+y}, \quad a>0 & \text{(extensions for some } x,\, y\text{)}, \\
(a^x)^y = a^{xy}, a>0 & \text{(extensions for some } x,\, y\text{)},
\end{array}$$

$$a^{-x} = \frac{1}{a^x}, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 31.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

31.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

31.4 Special derivatives.

Power like:

$$\frac{d}{dx}\left(x^\alpha\right) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 31.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

31.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x)+\lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 31.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 31.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y=\varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 31.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

31.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x\,dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x\,dx = \frac{1}{\ln a}\cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x\,dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x\,dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x\,dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x\,dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x}\,dx = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x\,dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x\,dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x\,dx = \ln\cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x\,dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x}\,dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x}\,dx = 2\operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x}\,dx = \frac{1}{2}\ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

31.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

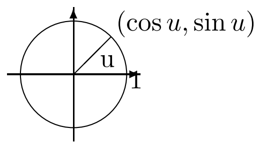

Figure 31.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \,\mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 31.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \,\mathrm{i}\,\sin(3u) &= (\cos u + \,\mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \,\mathrm{i}^3 \sin^3 u\\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \,\mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \,\mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamondsuit$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

31.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

31.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \ \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \ \mathrm{i}\,\sin x.$$

31.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

31.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,
2) power functions,
3) exponential functions,
4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^{\beta}}{x^{\alpha}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^{\alpha}}{a^{x}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^{\alpha} \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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