Problems, Theory and Solutions in Linear Algebra

Part 1 Euclidean Space

Marianna Euler; Norbert Euler

MARIANNA EULER AND
NORBERT EULER

PROBLEMS, THEORY AND SOLUTIONS IN LINEAR ALGEBRA

PART 1 EUCLIDEAN SPACE

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Preface

This book is the first part of a three-part series titled *Problems, Theory and Solutions in Linear Algebra*. This first part treats vectors in Euclidean space as well as matrices, matrix algebra and systems of linear equations. We solve linear systems by the use of Gauss elimination and by other means, and investigate the properties of these systems in terms of vectors and matrices. In addition, we also study linear transformations of the type $T: \mathbb{R}^n \to \mathbb{R}^m$ and derive the standard matrices that describe these transformations.

The second part in this series is subtitled *General Vector Spaces*. In this part we define a general vector space and introduce bases, dimensions and coordinates for these spaces. This gives rise to the coordinate mapping and other linear transformations between general vector spaces and Euclidean spaces. We also discuss several Euclidean subspaces, e.g., the null space and the column space, as well as eigenspaces of matrices. We then make use of the eigenvectors and similarity transformations to diagonalize square matrices.

In the third part, subtitled *Inner Product Spaces*, we include the operation of inner products for pairs of vectors in general vector spaces. This makes it possible to define orthogonal and orthonormal bases, orthogonal complement spaces and orthogonal projections of vectors onto finite dimensional subspaces. The so-called least squares solutions are also introduced here, as the best approximate solutions for inconsistent linear systems $A\mathbf{x} = \mathbf{b}$.

The aim of this series it to provide the student with a well-structured and carefully selected set of solved problems as well as a thorough revision of the material taught in a course in linear algebra for undergraduate engineering and science students. In each section we give a short summary of the most important theoretical concepts relevant to that section as **Theoretical Remarks**. This is followed by a variety of **Problems** that address these concepts. We then provide the complete **Solutions** of the stated problems. This is the structure throughout every book in this series. In each chapter an extensive list of exercises (with answers), that are similar to the solved problems treated in that particular chapter, are given.

Given the struture of the books in this series, it should be clear that the books are not traditional textbooks for a course in linear algebra. Rather, we believe that this series may serve as a supplement to any of the good undergraduate textbook in linear algebra. Our main goal is to guide the student in his/her studies by providing carefully selected solved problems and exercises to bring about a better understanding of the abstract notions in linear algebra, in particular for engineering and science students. The books in this series should also be helpful to develope or improve techniques and skills for problem solving. We foresee that students will find here alternate procedures, statements and exercises that are beyond some of the more traditional study material in linear algebra, and we hope that this will make the subject more interesting for the students.

A note to the Student

Our suggestion is that you first tackle the **Problems** yourself, if necessary with the help of the given **Theoretical Remarks**, before you look at the **Solutions** that are provided. In our opinion, this way of studying linear algebra is helpful, as you may be able to make new connections between statements and possibly learn some alternate ways of solving specific problems in linear algebra.

Each section in each chapter of this book (which constitutes Part 1 in this three-part series on linear algebra) is mostly self-contained, so you should be able to work with the problems of different sections in any order that you may prefer. Therefore, you do not need to start with Chapter 1 and work through all material in order to use the parts that appear, for example, in the last chapter.

To make it easier for you to navigate in this book we have, in addition to the usual Contents list at the beginning and the Index at the back of the book, also made use of colours to indicate the location of the **Theoretical Remarks**, the **Problems** and the **Solutions**.

This book includes over **100 solved problems** and more than **100 exercises** with answers. **Enjoy!**

Marianna Euler and Norbert Euler *Luleå, April 2016*

Mathematical symbols

$\mathbb{R}$:	The set of all real numbers.
$\mathbb{R}^n$:	The Euclidean space that contains all n-component vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ for all $v_j \in \mathbb{R}$.
$\|\mathbf{v}\|$:	The norm (or length) of a vector.
$\hat{\mathbf{v}}$:	The direction vector of $\mathbf{v}$; $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.
$\overrightarrow{P_1P_2}$:	A vector in $\mathbb{R}^3$ with the direction from P_1 to P_2.
$\mathbf{u} \cdot \mathbf{v}$:	The dot product (scalar product) for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.
$\mathbf{u} \times \mathbf{v}$:	The cross product (vector product) for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$.
$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$:	The scalar triple product for three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.
$\text{proj}_{\mathbf{v}}\, \mathbf{u}$:	The orthogonal projection of vector $\mathbf{u}$ onto vector $\mathbf{v}$.
$\{\mathbf{e_1},\ \mathbf{e_2},\ \cdots,\ \mathbf{e_n}\}$:	The set of standard basis vectors for $\mathbb{R}^n$.
$A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}] = [a_{ij}]$:	An $m \times n$ matrix with columns $\mathbf{a_j} \in \mathbb{R}^m$, $j = 1, 2, \ldots, n$.
$I_n = [\mathbf{e_1}\ \mathbf{e_2}\ \cdots\ \mathbf{e_n}]$:	The $n \times n$ identity matrix with $\mathbf{e_j}$ standard basis vectors for $\mathbb{R}^n$.
$\det A$ or $\|A\|$:	The determinant of the square matrix A.
A^{-1} :	The inverse of the square matrix A.
$A \sim B$:	The matrices A and B are row equivalent.
$[A\ \mathbf{b}]$:	The augmented matrix corresponding $A\mathbf{x} = \mathbf{b}$.
$\text{span}\,\{\mathbf{u_1},\ \mathbf{u_2},\ \cdots, \mathbf{u_p}\}$:	The set of vectors spanned by the vectors $\{\mathbf{u_1},\ \mathbf{u_2},\ \cdots, \mathbf{u_p}\}$.

Mathematical symbols (continued)

$T:\ \mathbb{R}^n \to \mathbb{R}^m$:	A transformation T mapping vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$.
C_T :	The co-domain of the transformation T.
$\mathcal{D}_T$:	The domain of the transformation T.
R_T :	The range of the transformation T.
$T:\ \mathbf{x} \mapsto T(\mathbf{x})$:	A transformation T mapping vector $\mathbf{x}$ to $T(\mathbf{x})$.
$T:\ \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$:	A linear transformation T mapping vector $\mathbf{x}$ to $A\mathbf{x}$.
$T_2 \circ T_1$:	A composite transformation.
T^{-1} :	The inverse transformation of T.

Chapter 1

Vectors, lines and planes in $\mathbb{R}^3$

The aim of this chapter:

We treat vectors in the Euclidean space $\mathbb{R}^3$ and use the standard vector operations of *vector addition*, the *multiplication of vectors with scalars* (real numbers), the *dot product* between two vectors, and the *cross product* between two vectors, to calculate lengths, areas, volumes and orthogonal (perpendicular) projections of one vector onto another vector (or onto a line). We also use vectors to *parametrize lines* in $\mathbb{R}^3$ and to find the equation that describes a plane in $\mathbb{R}^3$. We show how to calculate the distance between a point and a line, between a point and a plane, between two planes, between a line and a plane, as well as the distance between two lines in $\mathbb{R}^3$.

1.1 Vector operations and the dot product

In this section we study basic vector operations, including the dot product (or scalar product), for vectors in $\mathbb{R}^3$. We apply this to calculate the length (or norm) of vectors, the distance and angle between two vectors, as well as the orthogonal projection of one vector onto another vector and the reflection of one vector about another vector.

Theoretical Remarks 1.1.

Consider three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$. Assume that the initial point of the vectors are at the origin $(0,0,0)$ and that their terminal points are at (u_1,u_2,u_3), (v_1,v_2,v_3) and (w_1,w_2,w_3) respectively, called the **coordinates** or the **components** of the vectors. These vectors are also known as **position vectors** for these points. We write

$$\mathbf{u} = (u_1,u_2,u_3), \quad \mathbf{v} = (v_1,v_2,v_3), \quad \mathbf{w} = (w_1,w_2,w_3).$$

The position vector $\mathbf{u}$ for the point P with the coordinates (u_1,u_2,u_3) is shown in Figure 1.1. As a short notation, we indicate the coordinates of point P by $P:\ (u_1,u_2,u_3)$. The addition of the vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is another vector in $\mathbb{R}^3$, namely

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3).$$

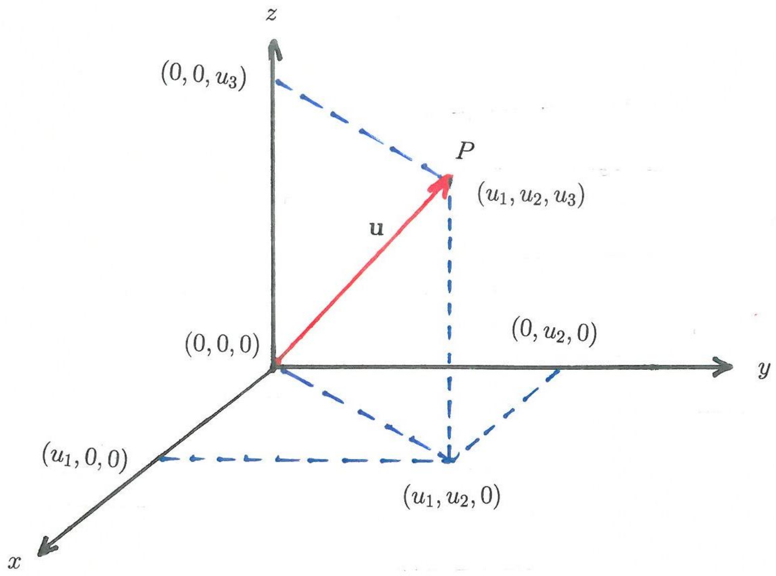

Figure 1.1: Position vector $\mathbf{u}$ of point P with coordinates (u_1, u_2, u_2).

See Figure 1.2. Given a third vector $\mathbf{w} \in \mathbb{R}^3$ we have the property

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

Multiplication of $\mathbf{u}$ with a real constant (or scalar) r, denoted by $r\mathbf{u}$, is another vector in $\mathbb{R}^3$, namely

$$r\mathbf{u} = (ru_1, ru_2, ru_3).$$

The vector $r\mathbf{u}$ is also called the **scaling** of $\mathbf{u}$ by r or the **dilation** of $\mathbf{u}$ by r. We have the following

Properties:

$$0\mathbf{u} = \mathbf{0} = (0,0,0) \quad \text{called the zero vector}$$

$$-\mathbf{u} = (-1)\mathbf{u} = (-u_1, -u_2, -u_3) \quad \text{called the negative of } \mathbf{u}$$

$$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-1)\mathbf{v} = (u_1-v_1, u_2-v_2, u_3-v_3)$$

$$\mathbf{u}-\mathbf{u} = \mathbf{0}.$$

The **dot product** (also known as the **Euclidean inner product** or the **scalar product**) of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\cdot\mathbf{v}$, is a real number defined as follows:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \in \mathbb{R}.$$

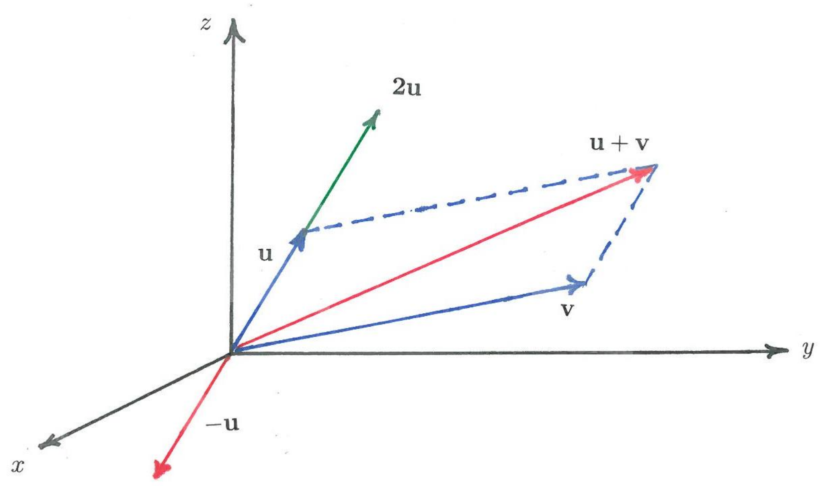

Figure 1.2: Addition of vectors **u** and **v**, as well as some scalings of vector **u**.

The **norm** of **u**, denoted by $\|\mathbf{u}\|$, is the length of **u** given by

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} \geq 0.$$

The **distance** between two points P_1 and P_2, with position vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ respectively, is given by the norm of the vector $\overrightarrow{P_1P_2}$ (see Figure 1.3), i.e.

$$\|\overrightarrow{P_1P_2}\| = \|\mathbf{v} - \mathbf{u}\| \geq 0.$$

A **unit vector** is a vector with norm 1. Every non-zero vector $\mathbf{u} \in \mathbb{R}^3$ can be **normalized** into a unique unit vector, denoted by $\hat{\mathbf{u}}$, which has the direction of **u**. That is, $\|\hat{\mathbf{u}}\| = 1$. This vector $\hat{\mathbf{u}}$ is called the **direction vector** of **u**. We have $\mathbf{u} = \|\mathbf{u}\|\, \hat{\mathbf{u}}$.
The set of unit vectors,

$$\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\},\ \text{where } \mathbf{e_1} = (1, 0, 0),\ \mathbf{e_2} = (0, 1, 0),\ \mathbf{e_3} = (0, 0, 1)$$

is known as the **standard basis** for $\mathbb{R}^3$ and the vectors are the standard basis vectors. The vector $\mathbf{u} = (u_1, u_2, u_3)$ can then be written in the form

$$\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}.$$

Let θ be the angle between **u** and **v**. From the definition of the dot product and the cosine law, it follows that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\, \|\mathbf{v}\| \cos\theta \in \mathbb{R}.$$

This means that the vectors **u** and **v** are **orthogonal** to each other (or perpendicular to each other) if and only if

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

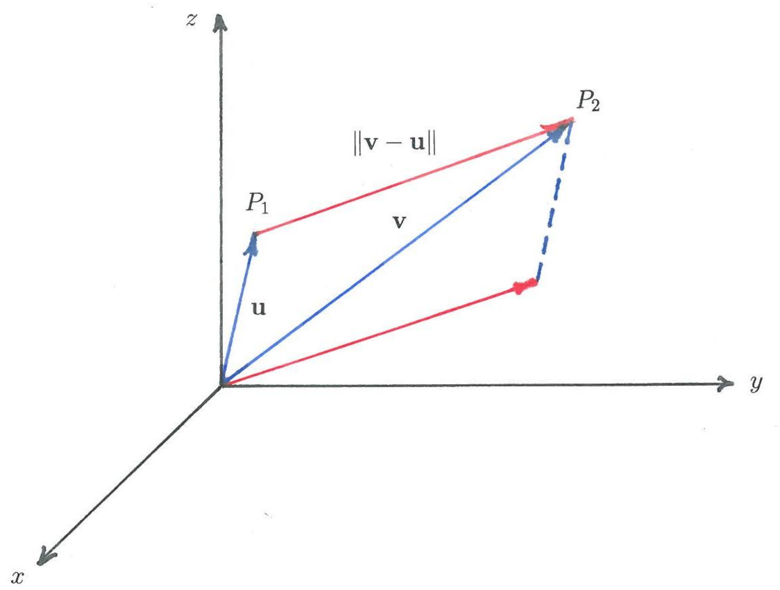

Figure 1.3: The distance between P_1 and P_2.

The **orthogonal projection** of $\mathbf{w}$ onto $\mathbf{u}$, denoted by $\text{proj}_{\mathbf{u}}\mathbf{w}$, is the vector

$$\text{proj}_{\mathbf{u}}\mathbf{w} = (\mathbf{w} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} \in \mathbb{R}^3,$$

where $\hat{\mathbf{u}}$ is the direction vector of $\text{proj}_{\mathbf{u}}\mathbf{w}$ and $|\mathbf{w} \cdot \hat{\mathbf{u}}|$ is the length of $\text{proj}_{\mathbf{u}}\mathbf{w}$ (note that $|\ |$ denotes the absolute value). See Figure 1.4.

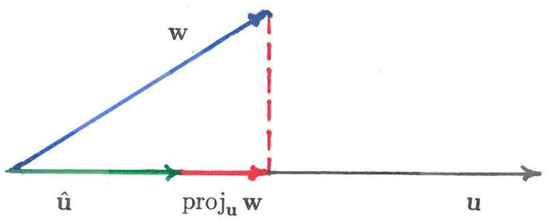

Figure 1.4: The orthogonal projection of $\mathbf{w}$ onto $\mathbf{u}$.

Problem 1.1.1.

Consider the following three vectors in $\mathbb{R}^3$: $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 0, 1)$, $\mathbf{w} = (3, 1, 0)$.

a) Find the length of $\mathbf{u}$ as well as the unit vector that gives the direction of $\mathbf{u}$.

b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

c) Project vector $\mathbf{w}$ orthogonally onto vector $\mathbf{v}$.

d) Find the vector that is the reflection of $\mathbf{w}$ about $\mathbf{v}$.

Solution 1.1.1.

a) The length of $\mathbf{u} = (1,2,3)$ is $\|\mathbf{u}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. The direction of $\mathbf{u} = (1,2,3)$ is given by the unit vector $\hat{\mathbf{u}}$, where

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{(1,2,3)}{\sqrt{14}} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}).$$

Note that $\|\hat{\mathbf{u}}\| = 1$.

b) The angle θ between $\mathbf{u} = (1,2,3)$ and $\mathbf{v} = (2,0,1)$ (See Figure 1.5) is calculated by the dot product

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta,$$

so that

$$\cos\theta = \frac{(1)(2)+(2)(0)+(3)(1)}{\sqrt{14}\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{14}}.$$

Hence

$$\theta = \cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{14}}\right).$$

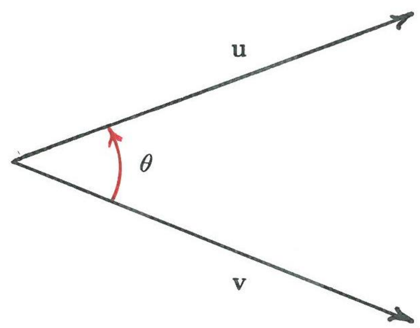

Figure 1.5: Angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$

c) The orthogonal projection of vector $\mathbf{w} = (3,1,0)$ onto vector $\mathbf{v} = (2,0,1)$, denoted by $\text{proj}_{\mathbf{v}}\,\mathbf{w}$, gives the component of vector $\mathbf{w}$ along the vector $\mathbf{v}$, also denoted by $\mathbf{w}_{\mathbf{v}}$. This orthogonal projection is

$$\text{proj}_{\mathbf{v}}\,\mathbf{w} = (\mathbf{w}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} = \left(\frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\ \mathbf{v} = \frac{(3)(2)+(1)(0)+(0)(1)}{2^2+0^2+1^2}(2,0,1) = (\frac{12}{5}, 0, \frac{6}{5}) = \mathbf{w}_{\mathbf{v}}.$$

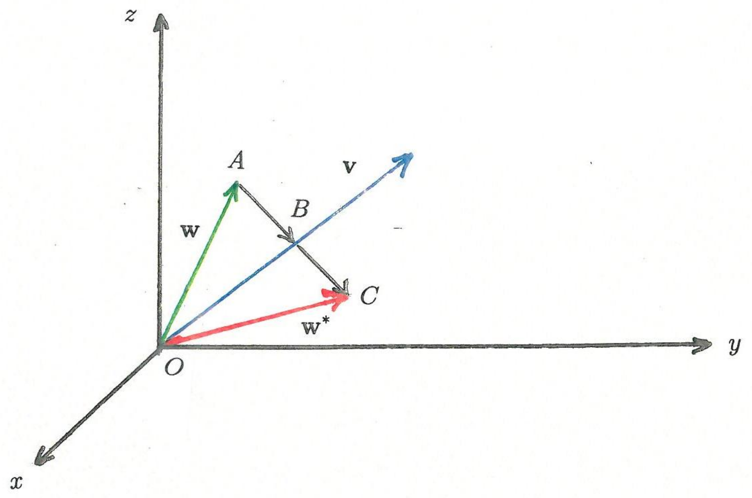

Figure 1.6: Vector $\mathbf{w}$ is reflected about $\mathbf{v}$

d) The reflection of $\mathbf{w}$ about $\mathbf{v}$ is given by vector $\mathbf{w}^*$ (see Figure 1.6), where

$$\mathbf{w}^* = \overrightarrow{OB} + \overrightarrow{BC}.$$

Since

$$\overrightarrow{OB} = \text{proj}_{\mathbf{v}}\mathbf{w}, \quad \overrightarrow{BC} = \overrightarrow{AB} \quad \text{and} \quad \overrightarrow{AB} = \text{proj}_{\mathbf{v}}\mathbf{w} - \mathbf{w}$$

we have

$$\mathbf{w}^* = \text{proj}_{\mathbf{v}}\mathbf{w} + (\text{proj}_{\mathbf{v}}\mathbf{w} - \mathbf{w}) = 2\,\text{proj}_{\mathbf{v}}\mathbf{w} - \mathbf{w}.$$

We calculate

$$\text{proj}_{\mathbf{v}}\mathbf{w} = (\frac{12}{5}, 0, \frac{6}{5})$$

$$\mathbf{w}^* = 2(\frac{12}{5}, 0, \frac{6}{5}) - (3, 1, 0) = (\frac{9}{5}, -1, \frac{12}{5}).$$

Problem 1.1.2.

Consider the following two vectors in $\mathbb{R}^3$: $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

a) Find the orthogonal projection of $\mathbf{u}$ onto the xy-plane.

b) Find the orthogonal projection of $\mathbf{u}$ onto the yz-plane.

c) Find the vector that is the reflection of $\mathbf{u}$ about the xz-plane.

d) Find the vector that results when $\mathbf{u}$ is first reflected about the xy-plane and then reflected about the xz-plane.

Solution 1.1.2.

a) The orthogonal projection of $\mathbf{u} = (u_1, u_2, u_3)$ onto the xy-plane is the vector $\mathbf{u}_{xy}$ which has zero z-component and the same x- and y-components as $\mathbf{u}$. Thus (see Figure 1.7)

$$\mathbf{u}_{xy} = (u_1, u_2, 0).$$

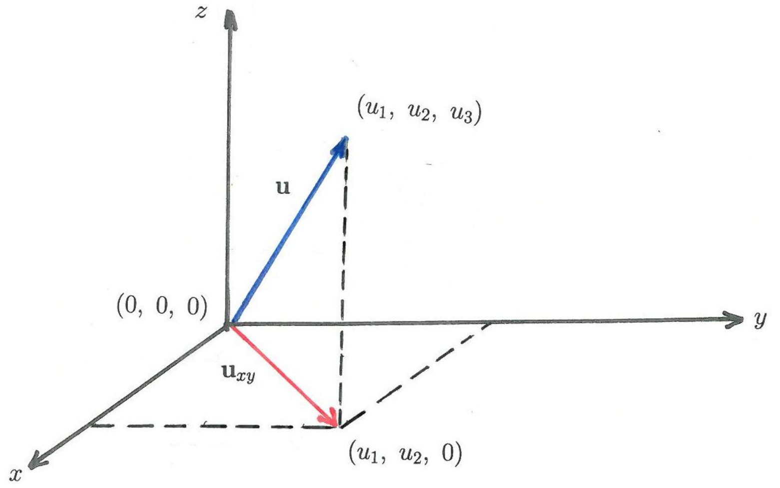

Figure 1.7: Orthogonal projection of $\mathbf{u}$ onto the xy-plane.

b) The orthogonal projection of $\mathbf{u} = (u_1, u_2, u_3)$ onto the yz-plane is the vector $\mathbf{u}_{yz}$, given by

$$\mathbf{u}_{yz} = (0, u_2, u_3).$$

c) The vector $\mathbf{u}^*{}_{xz}$, which is the reflection of $\mathbf{u} = (u_1, u_2, u_3)$ about the xz-plane, has the same x- and z-components as $\mathbf{u}$, but the negative y-component of $\mathbf{u}$. Thus

$$\mathbf{u}^*{}_{xz} = (u_1, -u_2, u_3).$$

d) We first reflect $\mathbf{u} = (u_1, u_2, u_3)$ about the xy-plane to obtain $\mathbf{u}^*{}_{xy} = (u_1, u_2, -u_3)$ and then we reflect $\mathbf{u}^*{}_{xy}$ about the xz-plane, which results in $(u_1, -u_2, -u_3)$.

1.2 The cross product

In this section we introduce the cross product (or vector product) between two vectors, as well as the scalar triple product between three vectors in $\mathbb{R}^3$. For example, the cross product between two vectors is used to find a third vector which is orthogonal to both these vectors in $\mathbb{R}^3$. We use these products to calculate, for example, the area of a parallelogram and the volume of a parallelepiped.

Theoretical Remarks 1.2.

Consider three vectors, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, in $\mathbb{R}^3$.

1) The **cross product** (also called **vector product**) of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is a vector in $\mathbb{R}^3$ which is defined as follows:

$$\mathbf{u} \times \mathbf{v} = (\|\mathbf{u}\| \, \|\mathbf{v}\| \sin \theta) \, \hat{\mathbf{e}} \in \mathbb{R}^3.$$

The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, where we have indicated the direction vector of $\mathbf{u} \times \mathbf{v}$ by $\hat{\mathbf{e}}$, so that $||\hat{\mathbf{e}}|| = 1$. The direction of $\hat{\mathbf{e}}$ is given by the right-handed triad and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. See Figure 1.8

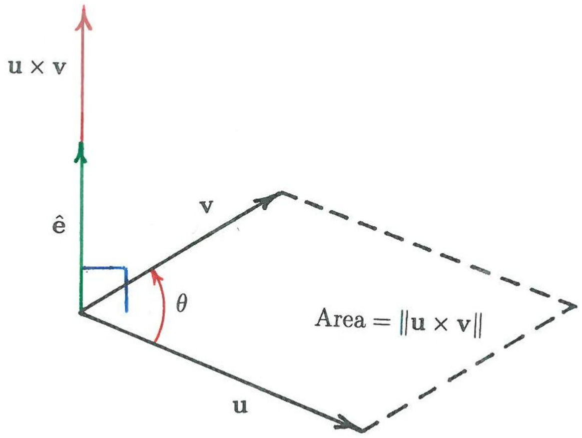

Figure 1.8: The cross product $\mathbf{u} \times \mathbf{v}$.

The cross product has the following

Properties:

a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

b) The norm $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram described by $\mathbf{u}$ and $\mathbf{v}$.

c) In terms of its coordinates, the cross product can be calculated by following the rule of calculations for determinants of 3×3 matrices (see Section 2.2. in Chapter 2), namely as follows

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \mathbf{e_1} \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \mathbf{e_2} \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \mathbf{e_3} \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{e_1} + (u_3v_1 - u_1v_3)\mathbf{e_2} + (u_1v_2 - u_2v_1)\mathbf{e_3}$$
$$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

Here

$$\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}, \quad \mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0), \quad \mathbf{e}_3 = (0,0,1),$$

is the **standard basis** for $\mathbb{R}^3$. The "$\det A$" denotes the **determinant** of a square matrix A. We also sometimes use the notation $|A|$ to denote the determinant of A, i.e. $\det A \equiv |A|$.

Remark: *The determinant of $n \times n$ matices is discussed in Chapter 2.*

d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0, \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0.$

e) Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

2) The product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \in \mathbb{R}$ is known as the **scalar triple product** and can be computed in terms of the determinant as follows:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

$$= (v_2w_3 - v_3w_2)u_1 + (v_3w_1 - v_1w_3)u_2 + (v_1w_2 - v_2w_1)u_3.$$

Then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}).$$

Consider a parallelepiped that is described by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. See Figure 1.9.

The volume of this parallelepiped is given by the absolute value of the scalar triple product of these three vectors. That is

$$\text{volume of parallelepiped} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \quad \text{cubic units.}$$

If the three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0.$$

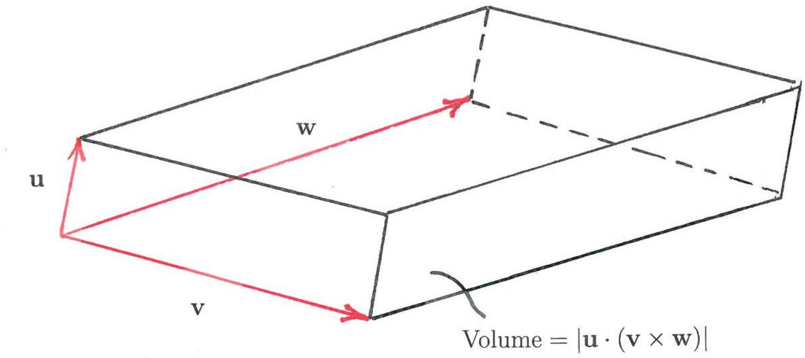

Figure 1.9: The parallelepiped described by **u**, **v** and **w**.

Problem 1.2.1.

Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (2, 0, 1), \quad \mathbf{w} = (3, 1, 0).$$

a) Find a vector that is orthogonal to both **u** and **v**.

b) Find the area of the parallelogram described by **u** and **v**.

c) Find the volume of the parallelepiped described by **u**, **v** and **w**.

Solution 1.2.1.

a) The vector $\mathbf{q} = \mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (2, 0, 1)$ (see Figure 1.10) and this cross product can be expressed in terms of the following determinant

$$\mathbf{q} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 2\mathbf{e_1} + 5\mathbf{e_2} - 4\mathbf{e_3} = (2, 5, -4).$$

Here $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is the standard basis for $\mathbb{R}^3$.

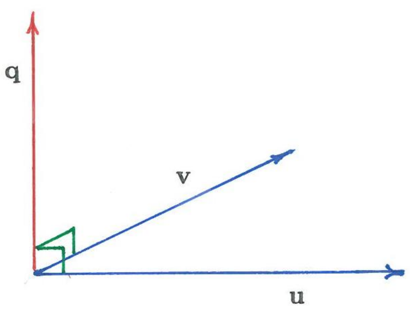

Figure 1.10: Vector **q** is orthogonal to both vectors **u** and **v**

b) The area of the parallelogram ABCD described by vectors **u** and **v** is given by $\|\mathbf{u} \times \mathbf{v}\|$. See Figure 1.11. In part a i) above we have calculated $\mathbf{u} \times \mathbf{v} = (2, 5, -4)$, so that

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{2^2 + 5^2 + (-4)^2} = 3\sqrt{5} \text{ square units.}$$

c) The volume of the parallelepiped described by vectors **u**, **v** and **w** is given by the absolute value of the scalar triple product, i.e.

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = | \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} |,$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

For the given vectors **u**, **v** and **w**, we obtain $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |11| = 11$ cubic units.

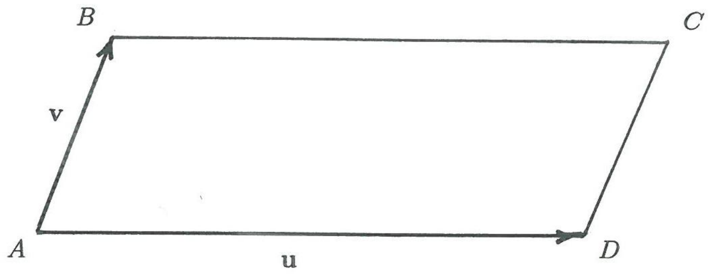

Figure 1.11: Parallelogram ABCD described by vectors **u** and **v**

Problem 1.2.2.

Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = (a, 2, -1), \quad \mathbf{u_2} = (4, 1, 0), \quad \mathbf{u_3} = (1, 5, -2),$$

where a is an unspecified real parameter.

a) Find the value(s) of a, such that the volume of the parallelepiped described by the given vectors $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ is one cubic units.

b) Find the area of each face of the parallelepiped which is described by the above given vectors $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ for $a = 0$.

Solution 1.2.2.

a) The volume of the parallelepiped is $V = |\mathbf{u_1} \cdot (\mathbf{u_2} \times \mathbf{u_3})|$ and we require that $V = 1$. Hence

$$V = |\begin{vmatrix} a & 2 & -1 \\ 4 & 1 & 0 \\ 1 & 5 & -2 \end{vmatrix}| = |-2a - 3| = 1,$$

so that $a = -1$ or $a = -2$.

b) The area of each face of the parallelepiped can be calculated as follows (see Figure 1.12):

$$\text{Area}_{\text{ face 1}} = \|\mathbf{u_1} \times \mathbf{u_3}\|, \quad \text{Area}_{\text{ face 2}} = \|\mathbf{u_2} \times \mathbf{u_3}\|, \quad \text{Area}_{\text{ face 3}} = \|\mathbf{u_1} \times \mathbf{u_2}\|,$$

all in square units.

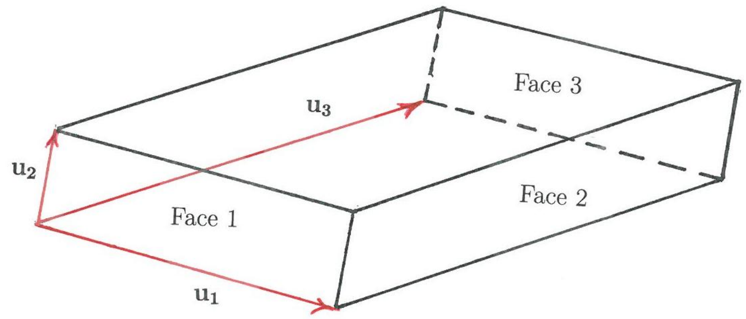

Figure 1.12: A parallelepiped described by $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$.

Now

$$\mathbf{u_1} \times \mathbf{u_3} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 0 & 2 & -1 \\ 1 & 5 & -2 \end{vmatrix} = \mathbf{e_1} - \mathbf{e_2} - 2\mathbf{e_3}$$

$$\mathbf{u_2} \times \mathbf{u_3} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 4 & 1 & 0 \\ 1 & 5 & -2 \end{vmatrix} = -2\mathbf{e_1} + 8\mathbf{e_2} + 19\mathbf{e_3}$$

$$\mathbf{u_1} \times \mathbf{u_2} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 0 & 2 & -1 \\ 4 & 1 & 0 \end{vmatrix} = \mathbf{e_1} - 4\mathbf{e_2} - 8\mathbf{e_3},$$

so that

$$\text{Area}_{\text{ face 1}} = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6} \quad \text{square units}$$

$$\text{Area}_{\text{ face 2}} = \sqrt{(-2)^2 + 8^2 + 19^2} = \sqrt{429} \quad \text{square units}$$

$$\text{Area}_{\text{ face 3}} = \sqrt{1^2 + (-4)^2 + (-8)^2} = 9 \quad \text{square units.}$$

1.3 Planes and their equations

In this section we describe planes in $\mathbb{R}^3$ and show how to derive their equations.

Theoretical Remarks 1.3.

1) The general **equation of a plane** in $\mathbb{R}^3$ is

$$ax + by + cz = d,$$

where a, b, c and d are given real numbers. All points (x, y, z) which lie on this plane must satisfy the equation of the plane, i.e. $ax + by + cz = d$.

2) The vector $\mathbf{n}$ with coordinates (a, b, c), i.e.

$$\mathbf{n} = (a, b, c),$$

is a vector that is orthogonal to the plane $ax + by + cz = d$. The vector $\mathbf{n}$ is known as the **normal vector** of the plane.

3) The equation of a plane can be calculated if three points that do not lie on the same line are given, or if the normal of the plane is known and one point on the plane is given.

Problem 1.3.1.

Consider the following three points in $\mathbb{R}^3$:

$$(1,2,3), \quad (2,0,1), \quad (3,1,0).$$

Find an equation of the plane Π that contains the given three points.

Solution 1.3.1.

Consider the points A, B and C with coordinates $(1,2,3)$, $(2,0,1)$ and $(3,1,0)$, respectively. Assume that these three points lie on the plane Π and that the point $P:\ (x,y,z,)$ is an arbitrary point on this plane. Consider now the vectors

$$\overrightarrow{AB} = (1,-2,-2), \quad \overrightarrow{AC} = (2,-1,-3), \quad \overrightarrow{AP} = (x-1, y-2, z-3).$$

Let $\mathbf{n}$ denote the normal to the plane Π. See Figure 1.13. Then

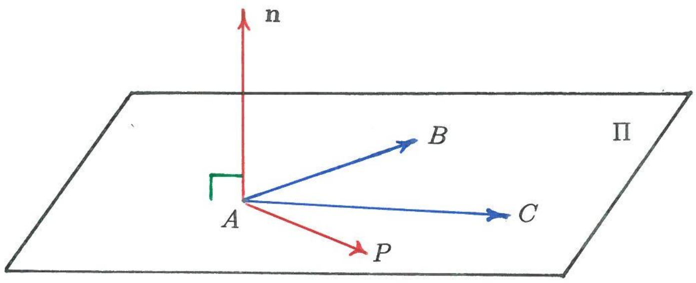

Figure 1.13: Plane Π with normal $\mathbf{n}$

$$\mathbf{n} = \overrightarrow{AC} \times \overrightarrow{AB} \qquad \text{and} \qquad \mathbf{n}\cdot\overrightarrow{AP} = 0,$$

so that

$$\mathbf{n} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 2 & -1 & -3 \\ 1 & -2 & -2 \end{vmatrix} = -4\mathbf{e_1} + \mathbf{e_2} - 3\mathbf{e_3} = (-4,1,-3).$$

The equation of the plane then follows from

$$0 = \mathbf{n}\cdot\overrightarrow{AP} = -4(x-1) + 1(y-2) - 3(z-3),$$

so that $4x - y + 3z = 11$.

Problem 1.3.2.

Consider four points in $\mathbb{R}^3$ with respective coordinates

$$(1,1,1), \quad (0,1,k), \quad (2,-1,-1) \text{ and } (-2,-1,1),$$

where k is an unspecified real parameter. Find the value(s) of k, such that the above four points lie in the same plane.

Solution 1.3.2.

Consider the four points

$$A:\ (1,1,1), \quad B:\ (0,1,k), \quad C:\ (2,-1,-1), \quad D:\ (-2,-1,1)$$

on a plane in $\mathbb{R}^3$. See Figure 1.14. Since the four point lie on the same plane we have

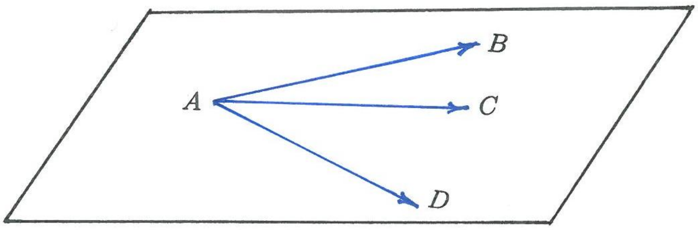

Figure 1.14: A plane that contains the points A, B, C and D.

$$\overrightarrow{AD} \cdot (\overrightarrow{AC} \times \overrightarrow{AB}) = 0$$

where

$$\overrightarrow{AD} = (-3,-2,0), \quad \overrightarrow{AC} = (1,-2,-2), \quad \overrightarrow{AB} = (-1,0,k-1)$$

and

$$\overrightarrow{AD} \cdot (\overrightarrow{AC} \times \overrightarrow{AB}) = \begin{vmatrix} -3 & -2 & 0 \\ 1 & -2 & -2 \\ -1 & 0 & k-1 \end{vmatrix} = 0.$$

Calculating the above determinant, we obtain the condition $8k - 12 = 0$, so that the value of k for which the four points lie on the same plane is

$$k = \frac{3}{2}.$$

Problem 1.3.3.

Find the equation of the plane in $\mathbb{R}^3$ that passes through the point $(1, 3, 1)$ and that is parallel to the plane

$$x + y - z = 1.$$

Solution 1.3.3.

We denote the given plane by Π_1, i.e.

$$\Pi_1 : \quad x + y - z = 1,$$

and denote by Π_2 the plane that we are seeking. See Figure 1.15. A normal vector for Π_1 is

$$\mathbf{n}_1 = (1, 1, -1)$$

and, since the plane Π_2 is parallel to the given plane Π_1, their normal vectors will also be parallel. Hence a normal vector $\mathbf{n_2}$ for Π_2 is the same as that of Π_1, namely

$$\mathbf{n}_2 = (1, 1, -1).$$

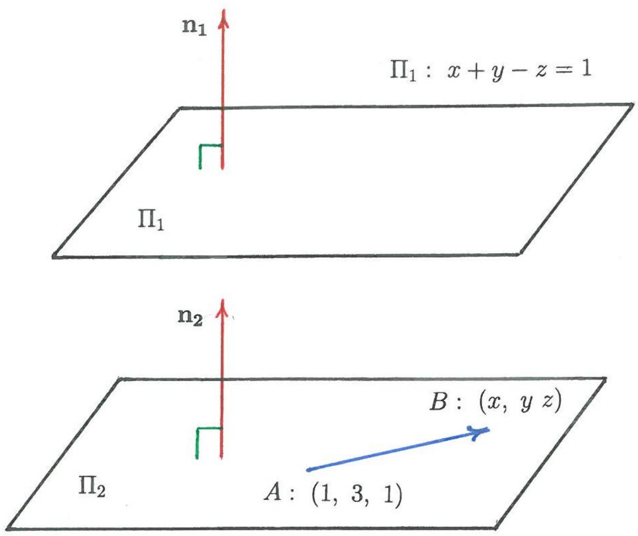

Figure 1.15: Two parallel planes Π_1 and Π_2

We know one point on the plane Π_2, namely the point $A : \ (1, 3, 1)$. Let B be an arbitrary point on the plane Π_2, say

$$B : \ (x, y, z).$$

Then vector $\overrightarrow{AB}$ takes the form

$$\overrightarrow{AB} = (x-1, y-3, z-1)$$

and this vector is orthogonal to the normal vector $\mathbf{n}_2$. Hence

$$\overrightarrow{AB} \cdot \mathbf{n}_2 = 0.$$

Upon evaluating the dot product $\overrightarrow{AB} \cdot \mathbf{n}_2$, we obtain

$$1(x-1) + 1(y-3) - 1(z-1) = 0.$$

The equation for Π_2 is therefore

$$\Pi_2 : \quad x + y - z = 3.$$

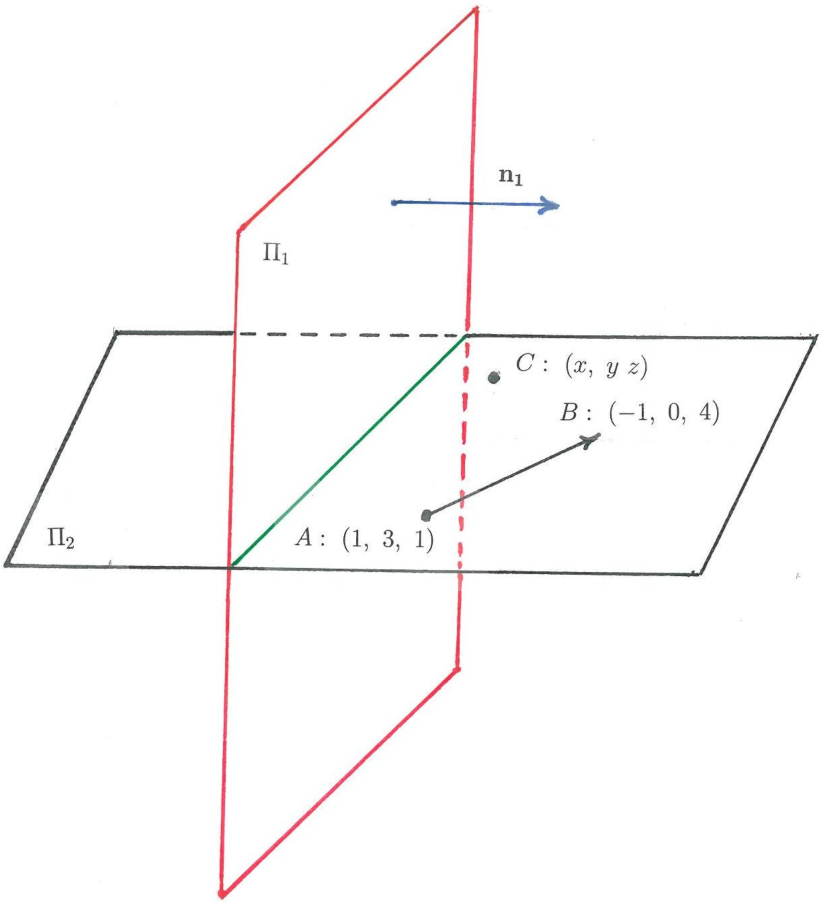

Figure 1.16: Two orthogonal planes Π_1 and Π_2

Problem 1.3.4.

Find the equation of the plane in $\mathbb{R}^3$ that passes through the points $(1,3,1)$ and $(-1,0,4)$ and that is orthogonal to the plane

$$x - y + 2z = 3.$$

Solution 1.3.4.

Let Π_1 be the given plane, i.e.

$$\Pi_1 : \quad x - y + 2z = 3$$

with normal vector

$$\mathbf{n}_1 = (1, -1, 2).$$

Let Π_2 be the plane that we are seeking. See Figure 1.16. We know two points on this plane, namely

$$A : \ (1,3,1), \qquad B : \ (-1,0,4).$$

In order to find the equation of Π_2 we first need to find its normal vector $\mathbf{n}_2$. Since Π_1 and Π_2 are orthogonal, it means that the normal vector $\mathbf{n}_2$ of Π_2 is orthogonal to every vector that is parallel to Π_2, say $\overrightarrow{AB}$, and $\mathbf{n}_2$ is orthogonal to $\mathbf{n}_1$. Thus

$$\mathbf{n}_2 = \mathbf{n}_1 \times \overrightarrow{AB},$$

where

$$\overrightarrow{AB} = (-2, -3, 3).$$

Hence

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & -1 & 2 \\ -2 & -3 & 3 \end{vmatrix} = 3\mathbf{e_1} - 7\mathbf{e_2} - 5\mathbf{e_3} = (3, -7, -5).$$

Let C be any point on Π_2, i.e.

$$C : \ (x, y, z).$$

The vector $\overrightarrow{AC}$ is orthogonal to the normal vector $\mathbf{n}_2$, so that

$$\mathbf{n}_2 \cdot \overrightarrow{AC} = 0,$$

where

$$\overrightarrow{AC} = (x-1, y-3, z-1).$$

Calculating the dot product $\mathbf{n_2} \cdot \overrightarrow{AC}$, we obtain

$$3(x-1) - 7(y-3) - 5(z-1) = 0,$$

so that the equation of the plane becomes

$$3x - 7y - 5z = -23.$$

1.4 Lines and their parametrizations

In this section we study lines ℓ in $\mathbb{R}^3$ and show how to derive parametic equations to describe ℓ. We derive a formula by which to calculate the distance from a point to a line and the distance between two lines. We also show how to project a vector orthogonally onto a line and how to reflect a vector about a line.

Theoretical Remarks 1.4.

The **parametric equation of a line** ℓ in $\mathbb{R}^3$ is of the form

$$\ell : \begin{cases} x = at + x_1 \\ y = bt + y_1 \\ z = ct + z_1 \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

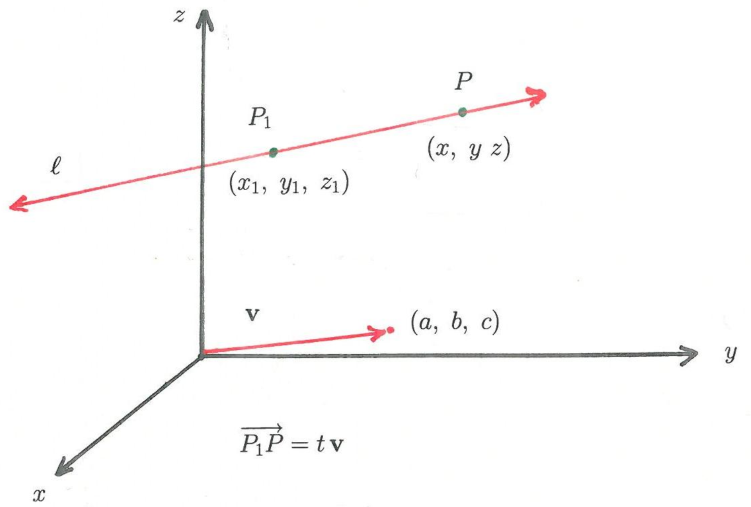

Figure 1.17: A line ℓ in $\mathbb{R}^3$

where (x_1, y_1, z_1) is a point on the line ℓ and $\mathbf{v} = (a, b, c)$ is the vector that is parallel to the line ℓ. See Figure 1.17. Here t is a parameter that can take on any real value. That is, for every point (x, y, z) on the line ℓ, there exists a unique value of t, such that

$$(x, y, z) = (at + x_1, bt + y_1, ct + x_1).$$

Problem 1.4.1.

Find a parametric equation of the line ℓ in $\mathbb{R}^3$, where $(-1, 1, 3)$ and $(2, 3, 7)$ are two points on ℓ.

a) Establish which of the following three points, if any, are on this line ℓ:
$(-4, -1, -1)$; $(-1, 2, 3)$; $(\frac{1}{2}, 2, 5)$

b) Is the vector $\mathbf{w} = (-6, -4, -8)$ parallel to the line ℓ? Explain.

Solution 1.4.1.

We are given two points that are on the line ℓ, namely $P_1 : (-1, 1, 3)$ and $P_2 : (2, 3, 7)$. Then the vector $\overrightarrow{P_1P_2}$ is parallel to ℓ and has the following coordinates:

$$\overrightarrow{P_1P_2} = (3, 2, 4).$$

Hence the vector $\mathbf{v}$ which is parallel to ℓ is

$$\mathbf{v} = \overrightarrow{P_1P_2} = (3, 2, 4).$$

Let $P : (x, y, z)$ be an arbitrary point on ℓ. Then

$$\overrightarrow{P_1P} = t\mathbf{v} \quad \text{or} \quad (x + 1, y - 1, z - 3) = t(3, 2, 4) \quad \text{for all } t \in \mathbb{R}.$$

Comparing the x-, $y-$, and z-components of the above vector equation, we obtain

$$x + 1 = 3t, \quad y - 1 = 2t, \quad z - 3 = 4t,$$

respectively. The parametric equation for ℓ is therefore

$$\ell : \begin{cases} x = 3t - 1 \\ y = 2t + 1 \\ z = 4t + 3 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

a) To find out whether the point $(-4, -1, -1)$ is on the line, we use the obtained parametic equation for ℓ and find t. That is, t must satisfy the relations

$$-4 = 3t - 1, \quad -1 = 2t + 1, \quad -1 = 4t + 3.$$

This leads to a unique solution for t, namely $t = -1$. Hence the point $(-4, -1, -1)$ is on ℓ.

For the point $(-1, 2, 3)$ we have

$$-1 = 3t - 1, \quad 2 = 2t + 1, \quad 3 = 4t + 3,$$

which cannot be satisfied for any value of t. Hence $(-1, 2, 3)$ is not a point on ℓ.

The point $(1/2, 2, 5)$ satisfies the parametric equation for $t = 1/2$, so that $(1/2, 2, 5)$ is a point on ℓ.

b) The vector $\mathbf{w} = (-6, -4, -8)$ is indeed parallel to the line ℓ, since

$$\mathbf{w} = -2\mathbf{v},$$

where $\mathbf{v}$ is parallel to ℓ.

Problem 1.4.2.

Find a parametric equation of the line ℓ in $\mathbb{R}^3$ which passes through the point $(1, -1, 2)$ and which is orthogonal to the lines ℓ_1 and ℓ_2, given in parametric form by

$$\ell_1 : \begin{cases} x = 2t \\ y = t \\ z = t - 1 \quad \text{for all } t \in \mathbb{R}, \end{cases} \qquad \ell_2 : \begin{cases} x = -3t + 1 \\ y = 2t \\ z = 4t - 1 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Solution 1.4.2.

Vector $\mathbf{v_1} = (2, 1, 1)$ is a vector that is parallel to ℓ_1 and $\mathbf{v_2} = (-3, 2, 4)$ is a vector that is parallel to ℓ_2. A vector that is orthogonal to both ℓ_1 and ℓ_2 is therefore

$$\mathbf{v} = \mathbf{v_1} \times \mathbf{v_2}$$

and this vector $\mathbf{v}$ is thus parallel to the line ℓ that we are seeking. We calculate $\mathbf{v}$:

$$\mathbf{v} = \mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 2 & 1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = 2\mathbf{e_1} - 11\mathbf{e_2} + 7\mathbf{e_3} = (2, -11, 7).$$

A parametric equation for the line ℓ is therefore

$$\ell : \begin{cases} x = 2t + 1 \\ y = -11t - 1 \\ z = 7t + 2 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Problem 1.4.3.

Consider a line ℓ in $\mathbb{R}^3$ with the following parametric equation

$$\ell : \begin{cases} x = at + x_1 \\ y = bt + y_1 \\ z = ct + z_1 \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

where (x_1, y_1, z_1) is a point on ℓ and

$$\mathbf{v} = (a, b, c)$$

is a vector parallel to ℓ.

a) Assume that the point $P_0 : (x_0, y_0, z_0)$ is not on the line ℓ. Find a formula for the distance from the point P_0 to ℓ.

b) Find the distance from the point $(-2, 1, 3)$ to the line ℓ, given by the parametric equation

$$\ell : \begin{cases} x = t + 1 \\ y = 3t - 4 \\ z = 5t + 2 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Solution 1.4.3.

a) We find a formula for the distance from the point $P_0(x_0, y_0, z_0)$ to the line ℓ, where

$$\ell : \begin{cases} x = at + x_1 \\ y = bt + y_1 \\ z = ct + z_1 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Here $P_1 : (x_1, y_1, z_1)$ is a point on the line ℓ. Let $s = \|\overrightarrow{P_0P_2}\|$ denote the distance from P_0 to ℓ, where P_2 is a point on ℓ which is not known. We consider the right-angled triangle $\triangle P_1P_2P_0$. See Figure 1.18.

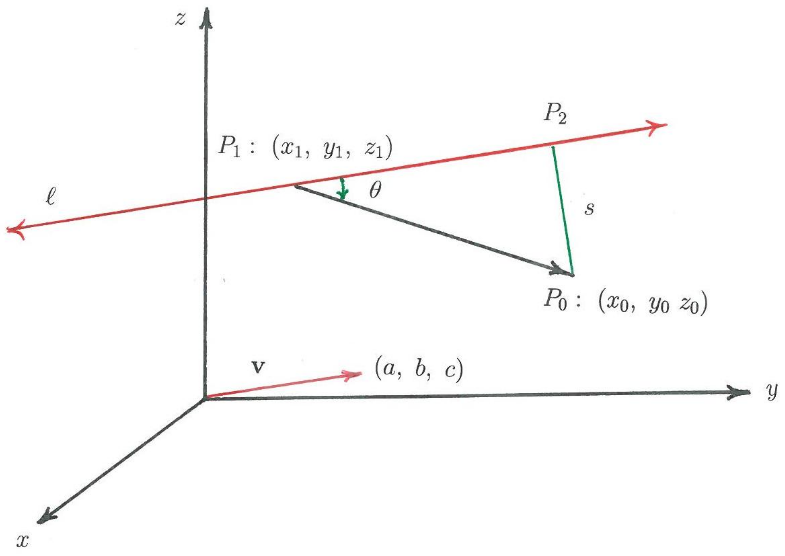

Figure 1.18: The distance from a point $P_0 : (x_0, y_0, z_0)$ to the line ℓ in $\mathbb{R}^3$

It follows that

$$s = \|\overrightarrow{P_0P_2}\| = \|\overrightarrow{P_1P_0}\| \sin\theta. \tag{1.4.1}$$

On the other hand we have, from the definition of the cross product, that

$$\|\overrightarrow{P_1P_0} \times \mathbf{v}\| = \|\overrightarrow{P_1P_0}\|\,\|\mathbf{v}\|\,\sin\theta. \tag{1.4.2}$$

Solving $||\overrightarrow{P_1P_0}||\,\sin\theta$ from (1.4.2) and inserting it into (1.4.1), we obtain the following formula for the distance:

$$s = \frac{\|\overrightarrow{P_1P_0} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

b) The given line ℓ passes through the point $P_1 : (1, -4, 2)$ and is parallel to the vector $\mathbf{v} = (1, 3, 5)$. Thus for the point $P_0 : (-2, 1, 3)$, we have

$$\overrightarrow{P_1P_0} = (-3, 5, 1)$$

and

$$\overrightarrow{P_1P_0} \times \mathbf{v} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ -3 & 5 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 22\mathbf{e_1} + 16\mathbf{e_2} - 14\mathbf{e_3} = (22, 16, -14).$$

Calculating the lengths of the vectors $\overrightarrow{P_1P_0} \times \mathbf{v}$ and $\mathbf{v}$, we obtain

$$\|\overrightarrow{P_1P_0} \times \mathbf{v}\| = \sqrt{(22)^2 + (16)^2 + (-14)^2} = 6\sqrt{26}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35},$$

so that the distance from the point P_0 to the given line ℓ is

$$s = \frac{\|\overrightarrow{P_1P_0} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{6\sqrt{26}}{\sqrt{35}}.$$

Problem 1.4.4.

Find a formula for the distance between two lines in $\mathbb{R}^3$ and use your formula to find the distance between the following two lines:

$$\ell_1 : \begin{cases} x = 2t + 1 \\ y = t - 1 \\ z = 3t + 1 \quad \text{for all } t \in \mathbb{R}, \end{cases} \qquad \ell_2 : \begin{cases} x = t \\ y = 2t + 2 \\ z = 1 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Solution 1.4.4.

Assume that $P_1 : (x_1, y_1, z_1)$ is a point on the line ℓ_1 and that $P_2 : (x_2, y_2, z_2)$ is a point on another line ℓ_2. Let $\mathbf{v_1}$ denote a vector that is parallel to ℓ_1 and $\mathbf{v_2}$ a vector that is parallel to ℓ_2 (see Figure 1.19). Now $\mathbf{v} = \mathbf{v_1} \times \mathbf{v_2}$ is a vector that is orthogonal to both

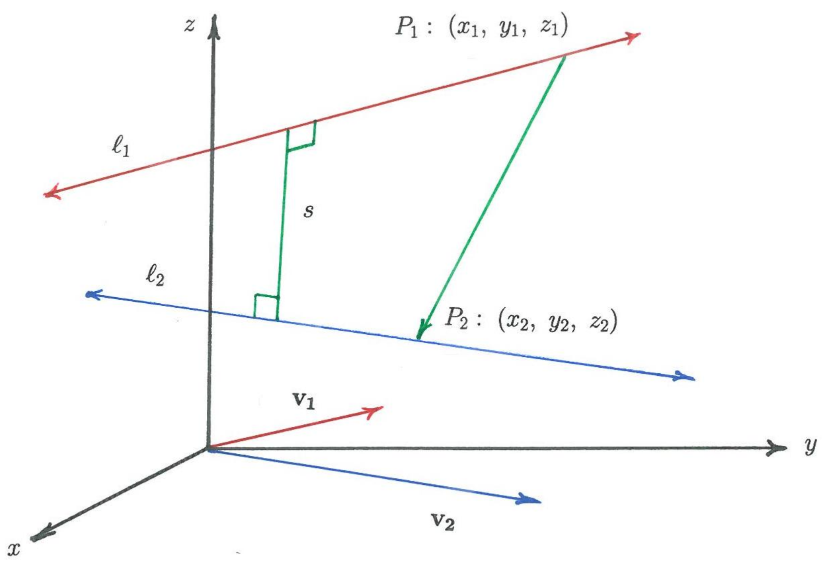

Figure 1.19: Distance s between two lines in $\mathbb{R}^3$

$\mathbf{v_1}$ and $\mathbf{v_2}$, and therefore $\mathbf{v}$ is orthogonal to the lines ℓ_1 and ℓ_2. To find the distance s between ℓ_1 and ℓ_2, we project $\overrightarrow{P_1P_2}$ orthogonally onto the vector $\mathbf{v}$. This leads to

$$s = \left\|\text{proj}_{\mathbf{v}}\overrightarrow{P_1P_2}\right\| = \frac{|\overrightarrow{P_1P_2} \cdot \mathbf{v}|}{\|\mathbf{v}\|} = \frac{|\overrightarrow{P_1P_2} \cdot (\mathbf{v_1} \times \mathbf{v_2})|}{\|\mathbf{v_1} \times \mathbf{v_2}\|}.$$

For the given line ℓ_1 we have $\mathbf{v_1} = (2, 1, 3)$ with a point $P_1 : (1, -1, 1) \in \ell_1$ and for the given line ℓ_2 we have $\mathbf{v_2} = (1, 2, 0)$ with a point $P_2 : (0, 2, 1) \in \ell_2$. Thus

$$\overrightarrow{P_1P_2} = (-1, 3, 0), \qquad \mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{vmatrix} = -6\mathbf{e_1} + 3\mathbf{e_2} + 3\mathbf{e_3} = (-6, 3, 3),$$

so that the distance s between ℓ_1 and ℓ_2 is

$$s = \frac{|(-1, 3, 0) \cdot (-6, 3, 3)|}{\sqrt{36 + 9 + 9}} = \frac{5}{\sqrt{6}}.$$

Problem 1.4.5.

Consider the following two vectors in $\mathbb{R}^3$:

$$\mathbf{u} = (-1, 3, 3), \quad \mathbf{v} = (2, -1, 4).$$

Consider now the line ℓ in $\mathbb{R}^3$, such that ℓ contains the point $(2, -1, 4)$ and the zero-vector $\mathbf{0} = (0, 0, 0)$.

a) Find the orthogonal projection of the vector $\mathbf{u}$ onto the line ℓ, i.e. calculate

$$\text{proj}_\ell\, \mathbf{u}.$$

b) Find the distance between the point $(-1, 3, 3)$ and the line ℓ.

c) Find the reflection of the vector $\mathbf{u}$ about the line ℓ.

Solution 1.4.5.

a) We aim to obtain the vector $\mathbf{w}$ which is the orthogonal projection of the vector $\mathbf{u}$ onto the line ℓ, i.e. $\mathbf{w}$ =proj$_\ell\, \mathbf{u}$. This can be achieved by projection $\mathbf{u}$ onto any position vector that is lying on this line ℓ, for example vector $\mathbf{v}$. See Figure 1.20. Thus

$$\mathbf{w} = \text{proj}_\ell \mathbf{u} = \text{proj}_\mathbf{v} \mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \quad \text{where } \hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}.$$

For $\mathbf{u} = (-1, 3, 3)$ and $\mathbf{v} = (2, -1, 4)$, we have

$$\mathbf{w} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \frac{(-1)(2) + (3)(-1) + (3)(4)}{2^2 + (-1)^2 + 4^2}(2, -1, 4) = \frac{1}{3}(2, -1, 4).$$

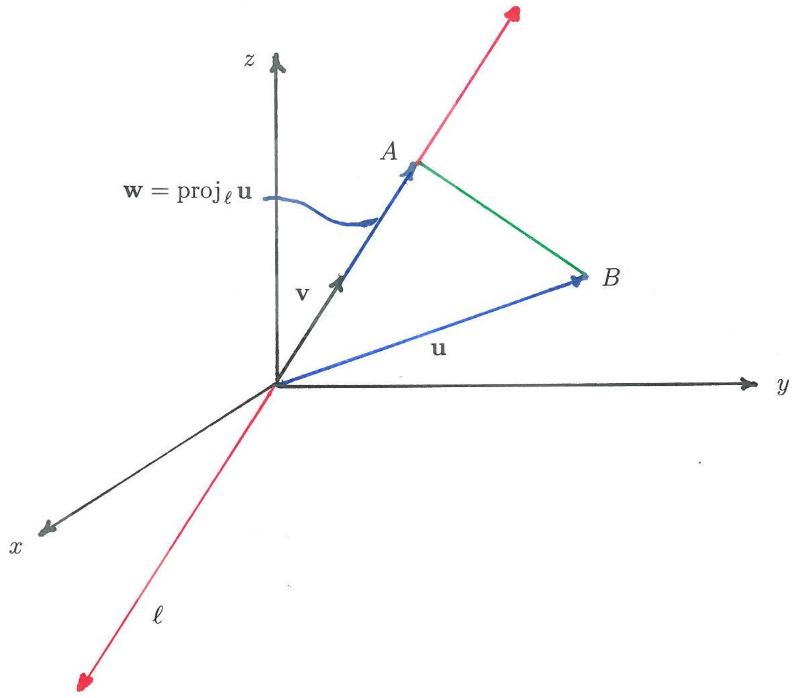

Figure 1.20: The orthogonal projection of $\mathbf{u}$ onto ℓ.

b) The distance d between the point $(-1, 3, 3)$ and the line ℓ is $\|\overrightarrow{AB}\|$ (see Figure 1.20). By vector addition we then have

$$\overrightarrow{AB} = \mathbf{u} - \mathbf{w} = (-1, 3, 3) - (\frac{2}{3}, -\frac{1}{3}, \frac{4}{3}) = (-\frac{5}{3}, \frac{10}{3}, \frac{5}{3}).$$

Thus

$$\|\overrightarrow{AB}\| = \sqrt{\frac{25}{9} + \frac{100}{9} + \frac{25}{9}} = \frac{5}{3}\sqrt{6}.$$

c) The reflection of the vector $\mathbf{u}$ about the line ℓ is given by the vector $\overrightarrow{OC}$. See Figure 1.21. By vector addition we have

$$\overrightarrow{OC} + \overrightarrow{CA} + \overrightarrow{AB} = \mathbf{u}.$$

However, $\overrightarrow{CA} = \overrightarrow{AB}$, so that

$$\overrightarrow{OC} = \mathbf{u} - 2\overrightarrow{AB},$$

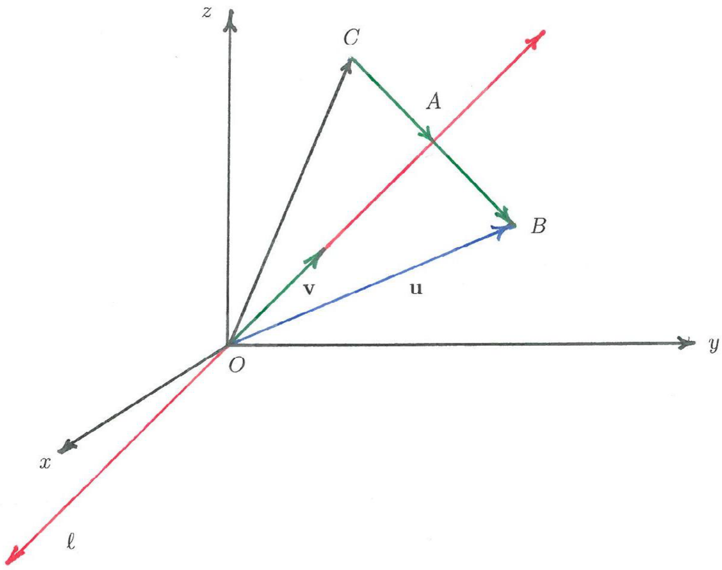

Figure 1.21: The reflection of **u** about ℓ.

where $\mathbf{u} = (-1, 3, 3)$ and $\overrightarrow{AB} = (-\frac{5}{3}, \frac{10}{3}, \frac{5}{3})$ (see part a) above). Thus the reflection about ℓ is

$$\overrightarrow{OC} = (-1, 3, 3) - 2(-\frac{5}{3}, \frac{10}{3}, \frac{5}{3}) = (\frac{7}{3}, -\frac{11}{3}, -\frac{1}{3}).$$

1.5 More on planes and lines

In this section we derive the distance between a point and a plane, as well as the distance between two planes. We also investigate the situation for a line that lies on a plane, a line that is projected orthogonally onto a plane, and a line that is reflected about a plane.

Theoretical Remarks 1.5.

1. Given the equation of a plane

   $$\Pi : \ ax + by + cz = d,$$

   the distance s from the point

   $$P_0 : \ (x_0, y_0, z_0)$$

   to the plane Π is

   $$s = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

2. Given two parallel planes, $\Pi_1 : \ ax + by + cz = d_1$ and $\Pi_2 : \ ax + by + cz = d_2$, the distance s between Π_1 and Π_2 is

   $$s = \frac{|d_1 - d_2|}{\|\mathbf{n}\|},$$

   where $\mathbf{n} = (a, b, c)$ is the normal vector for both planes.

Remark: *Any two planes in $\mathbb{R}^3$ that do not intersect must be parallel.*

Problem 1.5.1.

Consider a plane $ax + by + cz = d$ and a point $P_0 : (x_0, y_0, z_0)$, such that P_0 is not a point on this plane.

a) Find a formula for the distance from the point $P_0 : (x_0, y_0, z_0)$ to the plane $ax + by + cz = d$.

b) Find the distance from the point $(1, 2, 2)$ to the plane which passes through the origin $(0, 0, 0)$, as well as through the points $(1, 1, -1)$ and $(0, 2, 1)$.

Solution 1.5.1.

a) We consider the plane

$$\Pi: \; ax + by + cz = d$$

and a point

$$P_0 : (x_0, y_0, z_0) \notin \Pi.$$

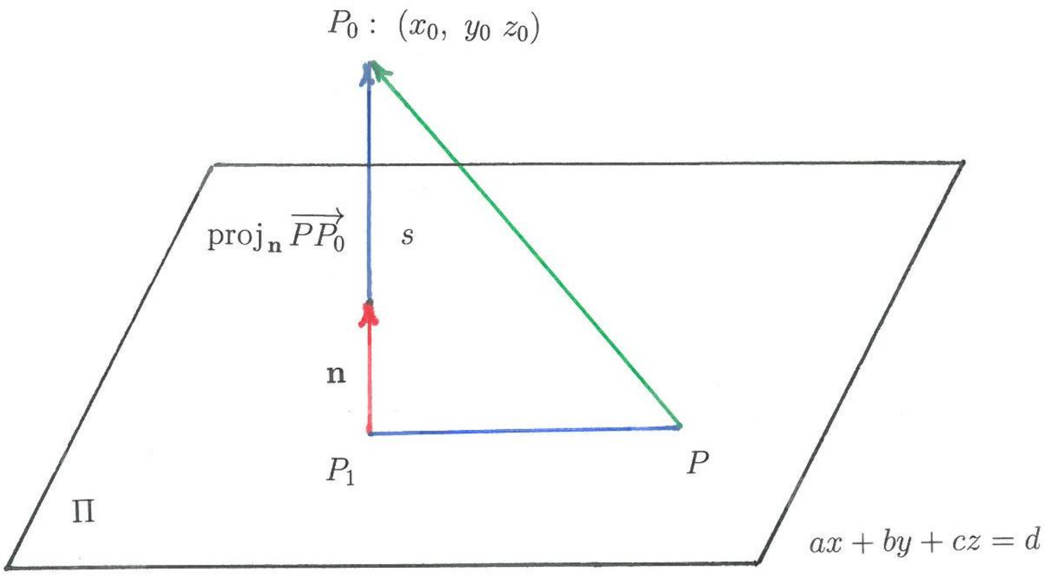

Figure 1.22: The point $P_0 : (x_0, y_0, z_0)$ and a plane $\Pi : \; ax + by + cz = d$ in $\mathbb{R}^3$

The normal vector $\mathbf{n}$ for the plane Π is $\mathbf{n} = (a, b, c)$ (see Figure 1.22). Consider now an arbitrary point on Π, say point $P : (x, y, z)$, and project vector $\overrightarrow{PP_0}$ orthogonally onto $\mathbf{n}$. This gives the distance s from the point P_0 to the plane Π, i.e.

$$s = \|\text{proj}_{\mathbf{n}} \overrightarrow{PP_0}\| = |\overrightarrow{PP_0} \cdot \hat{\mathbf{n}}| = \frac{|\overrightarrow{PP_0} \cdot \mathbf{n}|}{\|\mathbf{n}\|},$$

where $\hat{\mathbf{n}} = \mathbf{n}/\|\mathbf{n}\|$ and

$$\overrightarrow{PP_0} = (x_0 - x, y_0 - y, z_0 - z)$$

$$\begin{aligned}\overrightarrow{PP_0} \cdot \mathbf{n} &= a(x_0 - x) + b(y_0 - y) + c(z_0 - z)\\ &= -(ax + by + cz) + ax_0 + by_0 + cz_0\\ &= -d + ax_0 + by_0 + cz_0.\end{aligned}$$

Thus the distance from P_0 to Π is

$$s = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}, \qquad (1.5.1)$$

where $|\ \ |$ denotes the absolute value.

b) We seek the distance from $P_0 : (1, 2, 2)$ to the plane that contains the origin $O : (0, 0, 0)$, as well as the points $A : (1, 1, -1)$ and $B : (0, 2, 1)$. We name this plane Π. First we derive the equation of the plane Π.

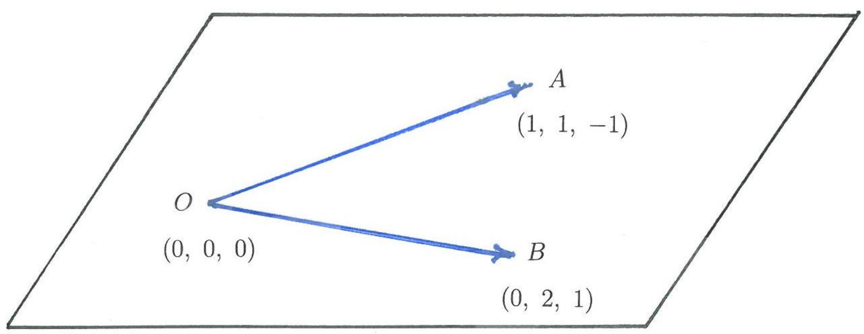

Figure 1.23: A plane in $\mathbb{R}^3$ that contains the points O, A and B.

Consider the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (see Figure 1.23). Then the normal vector $\mathbf{n}$ for Π is

$$\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OB},$$

where

$$\overrightarrow{OA} = (1, 1, -1), \qquad \overrightarrow{OB} = (0, 2, 1).$$

Calculating the above cross product, we obtain

$$\mathbf{n} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3\mathbf{e_1} - \mathbf{e_2} + 2\mathbf{e_3} = (3, -1, 2).$$

Since the plane Π passes through $O : (0, 0, 0)$, the equation for Π must be

$$3x - y + 2z = 0,$$

so that the distance s from the point $P_0 : (1, 2, 2)$ to the plane Π is

$$s = \frac{|(3)(1) + (-1)(2) + (2)(2)|}{\sqrt{9 + 1 + 4}} = \frac{5}{\sqrt{14}}.$$

Problem 1.5.2.

Find the distance between the planes $\Pi_1: \; 2x - 3y + 4z = 5$ and $\Pi_2: \; 4x - 6y + 8z = 16$.

Solution 1.5.2.

We are given two planes, namely $\Pi_1 : 2x - 3y + 4z = 5$ and $\Pi_2 : \; 4x - 6y + 8z = 16$. Dividing the equation of the plane Π_2 by 2 we obtain $2x-3y+4z = 8$. The normal vector $\mathbf{n}$ for both planes is therefore

$$\mathbf{n} = (2, -3, 4),$$

so that we can conclude that the two planes are parallel. We now choose any point $P_0 : \; (x_0, y_0, z_0)$ on Π_1 and then calculate the distance s from point P_0 to $\Pi_2/2$. We make use of the formula (1.5.1), namely

$$s = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

as derived in Problem 1.5.1 a) and given in Theoretical Remark 1.5, to calculate the distance from the point P_0 to the plane Π_2. For Π_2 we have $a = 2$, $b = -3$, $c = 4$ and $d = 8$. To find a point P_0 that lies on the plane Π_1, we let $x = 1$ and $y = 0$, and insert those values into the equation for Π_1 to calculate z. We obtain

$$2(1) - 3(0) + 4z = 5, \quad \text{so that } z = \frac{3}{4}.$$

Thus we have $P_0 : \; (1, 0, \frac{3}{4})$. Calculating s, we obtain

$$s = \frac{|(2)(1) - (3)(0) + (4)(3/4) - 8|}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{|-3|}{\sqrt{29}} = \frac{3}{\sqrt{29}}.$$

Alternatively, we can use the formula $s = |d_1 - d_2|/\|\mathbf{n}\|$ with $d_1 = 5$ and $d_2 = 8$, as given in Theoretical Remark 1.5.

Problem 1.5.3.

Consider the following line ℓ in $\mathbb{R}^3$:

$$\ell : \begin{cases} x = 2t + 1 \\ y = -2t + 1 \\ z = 6t - 6 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Find all real values for the parameter b, such that every point on ℓ is on the plane

$$\frac{2}{3}x + by + \frac{1}{9}z = 1.$$

Solution 1.5.3.

Since every point on ℓ must lie on the given plane, we insert x, y and z, given by the parametric equation for ℓ, into the equation of the plane. This leads to

$$\frac{2}{3}(2t+1) + b(-2t+1) + \frac{1}{9}(6t-6) = 1.$$

Simplifying and collecting coefficients of t in the above relation, we obtain

$$(18 - 18b)t + 9b - 9 = 0 \quad \text{for all } t \in \mathbb{R}.$$

We conclude that

$$18 - 18b = 0 \quad \text{and} \quad 9b - 9 = 0,$$

so that $b = 1$. Thus every point which is on ℓ is on the given plane, if and only if the plane has the equation

$$\frac{2}{3}x + y + \frac{1}{9}z = 1, \quad \text{or, equivalently, the equation} \quad 6x + 9y + z = 9.$$

Problem 1.5.4.

Consider the plane $\Pi:\ x + y - z = -3$ and the line

$$\ell : \begin{cases} x = t + 1 \\ y = 2t + 1 \\ z = 2t + 2 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

a) Find a parametric equation for the line $\hat{\ell}$, such that $\hat{\ell}$ is the orthogonal projection of the given line ℓ onto the given plane Π.

b) Find all points on the given line ℓ, such that the distance between those points and the given plane Π is $2\sqrt{3}$.

c) Find a parametric equation for the line ℓ^*, such that ℓ^* is the reflection of the given line ℓ about the given plane Π.

Solution 1.5.4.

a) First we find the intersection of the given line ℓ with the plane Π:
An arbitrary point P_t on ℓ has the coordinates

$$P_t:\ (t+1,\ 2t+1,\ 2t+2),$$

so that, for every $t \in \mathbb{R}$, P_t is a point on ℓ. To find the intersection of ℓ with Π, we insert

$$x = t + 1, \quad y = 2t + 1, \quad z = 2t + 2$$

into the equation of Π. This leads to

$$(t + 1) + (2t + 1) - (2t + 2) = -3,$$

from which we can solve t, to obtain $t = -3$. Therefore, the point P which lies on both ℓ and Π has the following coordinates (see Figure 1.24):

$$P : \ (-2, -5, -4).$$

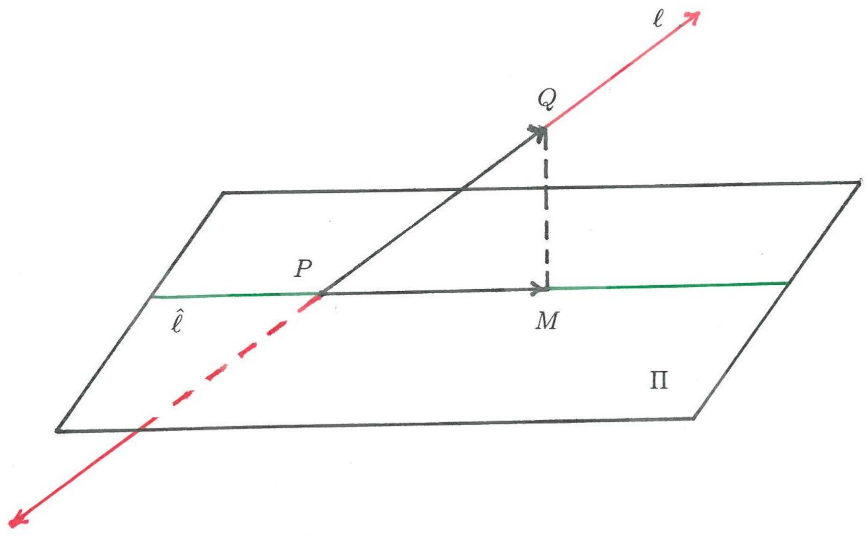

Figure 1.24: The line of orthogonal projection $\hat{\ell}$ of the line ℓ onto the plane Π.

To find the direction of $\hat{\ell}$, such that $\hat{\ell}$ is the line that represents the orthogonal projection of ℓ onto Π, we choose any point Q on ℓ (different from the point P), say the point

$$Q : \ (1, 1, 2).$$

Then we have $\overrightarrow{PQ} = (3, 6, 6)$ and following Figure 1.24, we obtain

$$\overrightarrow{PM} = \overrightarrow{PQ} - \overrightarrow{MQ},$$

where $\overrightarrow{PM}$ is the orthogonal projection of $\overrightarrow{PQ}$ onto $\hat{\ell}$ and hence $\overrightarrow{PM}$ is the orthogonal projection of $\overrightarrow{PQ}$ onto Π. To find $\overrightarrow{MQ}$ we project $\overrightarrow{PQ}$ orthogonally onto the normal vector $\mathbf{n}$ of Π, where $\mathbf{n} = (1, 1, -1)$. Thus

$$\overrightarrow{MQ} = \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} = (\overrightarrow{PQ} \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} = \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}$$

Calculating the above orthogonal projection we obtain $\overrightarrow{MQ} = (1, 1 - 1)$, so that

$$\overrightarrow{PM} = (3, 6, 6) - (1, 1, -1) = (2, 5, 7).$$

Since the line $\hat{\ell}$ is passing through the point $P: \ (-2, -5, -4)$ and has the direction $\overrightarrow{PM}$, the parametric equation for $\hat{\ell}$ is

$$\hat{\ell} : \begin{cases} x = 2t - 2 \\ y = 5t - 5 \\ z = 7t - 4 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

b) In **Problem 1.5.1** we have derived a formula for the distance s from the point $P_0 : (x_0, y_0, z_0)$ to the plane $ax + by + cz = d$, namely the formula

$$s = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Any point S_t on ℓ has the coordinates

$$S_t : \ (t + 1, \ 2t + 1, \ 2t + 2)$$

for any choice $t \in \mathbb{R}$. We can thus calculate the distance s from the point S_t to the given plane $\Pi : \ x + y - z = -3$ by using the above formula. We obtain

$$s = \frac{|1(t + 1) + 1(2t + 1) - 1(2t + 2) - (-3)|}{\sqrt{1 + 1 + 1}} = \frac{|t + 3|}{\sqrt{3}}.$$

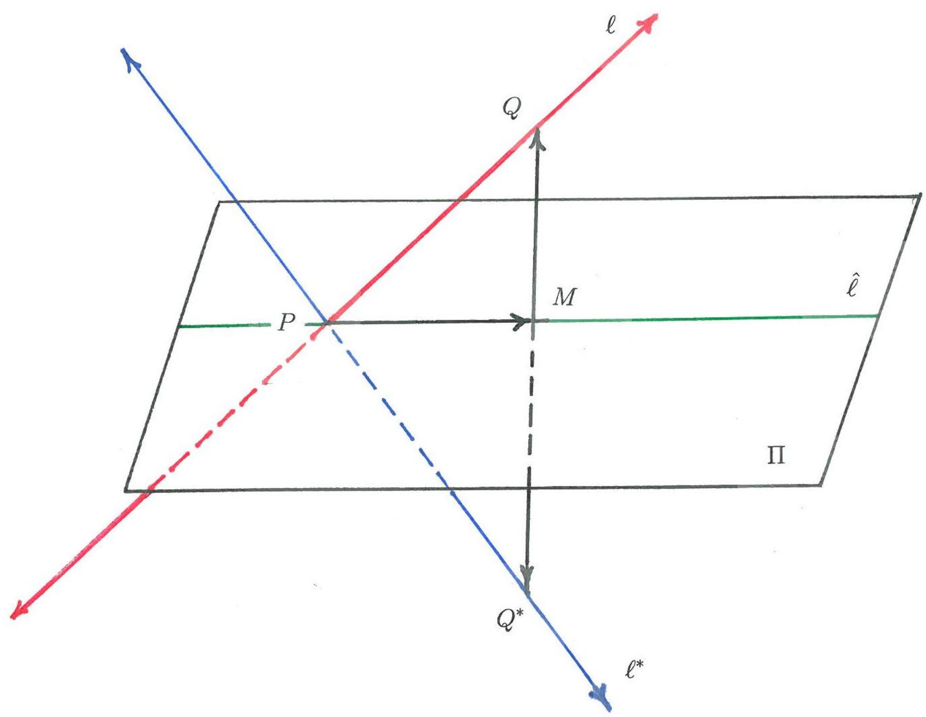

Figure 1.25: The line of reflection ℓ^* of the line ℓ about the plane Π.

We now seek the point S_t, such that $s = 2\sqrt{3}$. Hence we have

$$\frac{|t + 3|}{\sqrt{3}} = 2\sqrt{3} \quad \text{or} \quad |t + 3| = 6.$$

Thus $t = 3$ or $t = -9$. Using these two values of t for the coordinates of S_t, we obtain the following two points on ℓ which are a distance $2\sqrt{3}$ away from Π, namely the points with coordiantes

$$(4, 7, 8) \quad \text{and} \quad (-8, -17, -16).$$

c) We need to find a parametric equation for the line ℓ^*, such that ℓ^* is the reflection of the given line ℓ about the given plane Π. Clearly, ℓ^* can be obtained by finding the reflection of ℓ about the line $\hat{\ell}$, which has already been obtained in part a) above. Note also that, as given in part a), we have

$$P: \ (-2, -5, -4) \quad \text{and} \quad Q: \ (1, 1, 2).$$

Let Q^* denote the point on ℓ^*, such that

$$\overrightarrow{MQ^*} = -\overrightarrow{MQ}.$$

Following Figure 1.25, we have

$$\overrightarrow{PQ^*} = \overrightarrow{PM} + \overrightarrow{MQ^*},$$

where $\overrightarrow{PM} = (2, 5, 7)$ and $\overrightarrow{MQ^*} = -\overrightarrow{MQ} = (-1, -1, 1)$ [see part a) of this problem]. Thus

$$\overrightarrow{PQ^*} = (2, 5, 7) + (-1, -1, 1) = (1, 4, 8).$$

Since ℓ^* is passing through the point $P: \ (-2, -5, -4)$ and has the direction given by the vector $\overrightarrow{PQ^*} = (1, 4, 8)$, the parametric equation for ℓ^* takes the form

$$\ell^*: \begin{cases} x = t - 2 \\ y = 4t - 5 \\ z = 8t - 4 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

Problem 1.5.5.

Consider the plane $\Pi: \ 2x + y + z = 5$ and the line

$$\ell: \begin{cases} x = t + 2 \\ y = -5t + 1 \\ z = 3t + 3 \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

where Π and ℓ are parallel.

a) Find a parametric equation for the line $\hat{\ell}$, such that $\hat{\ell}$ is the orthogonal projection of the given line ℓ onto the given plane Π.

b) Find a parametric equation for the line ℓ, such that ℓ is the reflection of the given line ℓ about the given plane Π.

Solution 1.5.5.

a) We need to find the line $\hat{\ell}$ that is the orthogonal projection of the line ℓ, namely

$$\ell : \begin{cases} x = t + 2 \\ y = -5t + 1 \\ z = 3t + 3 \qquad \text{for all } t \in \mathbb{R} \end{cases}$$

onto the given plane $\Pi : \ 2x + y + z = 5$.

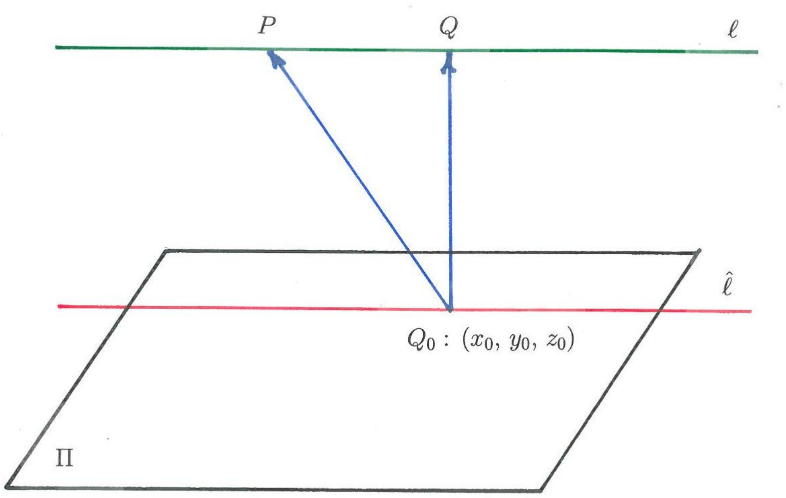

Figure 1.26: The orthogonal projection of ℓ onto Π

We refer to Figure 1.26 and choose any two points P and Q on ℓ by respectively setting $t = 0$ and $t = 1$ in the above parametric equation for ℓ. This leads to

$$P : \ (2, 1, 3), \qquad Q : \ (3, -4, 6).$$

We now seek the point $Q_0 = (x_0, y_0, z_0)$ in the plane Π, such that

$\overrightarrow{Q_0Q}$ is orthogonal to $\overrightarrow{QP}$

and

$\overrightarrow{Q_0Q}$ is parallel to $\mathbf{n} = (2, 1, 1)$,

where $\mathbf{n}$ is the normal of the plane Π. We have

$$\overrightarrow{Q_0Q} = (3 - x_0, -4 - y_0, 6 - z_0)$$
$$\overrightarrow{Q_0P} = (2 - x_0, 1 - y_0, 3 - z_0).$$

Moreover

$$\begin{aligned}\overrightarrow{Q_0Q} = \text{proj}_{\boldsymbol{n}}\, \overrightarrow{Q_0P} &= \left(\frac{\overrightarrow{Q_0P} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} \\ &= \left(\frac{2(2 - x_0) + 1(1 - y_0) + 1(3 - z_0)}{4 + 1 + 1}\right) (2, 1, 1) \\ &= \left(\frac{8 - (2x_0 + y_0 + z_0)}{6}\right) (2, 1, 1).\end{aligned}$$

Since Q_0 is a point on Π the coordinates of Q_0 have to satisfy the equations for Π, i.e.

$$2x_0 + y_0 + z_0 = 5,$$

so that

$$\text{proj}_{\boldsymbol{n}}\, \overrightarrow{Q_0P} = \left(\frac{8 - 5}{6}\right) (2, 1, 1) = (1, \frac{1}{2}, \frac{1}{2}).$$

Hence

$$(3 - x_0, -4 - y_0, 6 - z_0) = (1, \frac{1}{2}, \frac{1}{2}),$$

and by comparing the x-, y- and z-components, we obtain

$$x_0 = 2, \quad y_0 = -\frac{9}{2}, \quad z_0 = \frac{11}{2}, \text{ i.e. } Q_0 : \ (2, -\frac{9}{2}, \frac{11}{2}).$$

Clearly Q_0 is a point on $\hat{\ell}$, where $\hat{\ell}$ has the same direction vector $\mathbf{v}$ as ℓ, namely

$$\mathbf{v} = (1, -5, 3).$$

A parameteric equation for $\hat{\ell}$ is therefore

$$\hat{\ell}: \begin{cases} x = t + 2 \\ y = -5t - \dfrac{9}{2} \\ z = 3t + \dfrac{11}{2} \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

b) We need to find the line ℓ^* that is the reflection of the line ℓ, namely

$$\ell: \begin{cases} x = t + 2 \\ y = -5t + 1 \\ z = 3t + 3 \qquad \text{for all } t \in \mathbb{R} \end{cases}$$

about the given plane $\Pi: \; 2x + y + z = 5$.

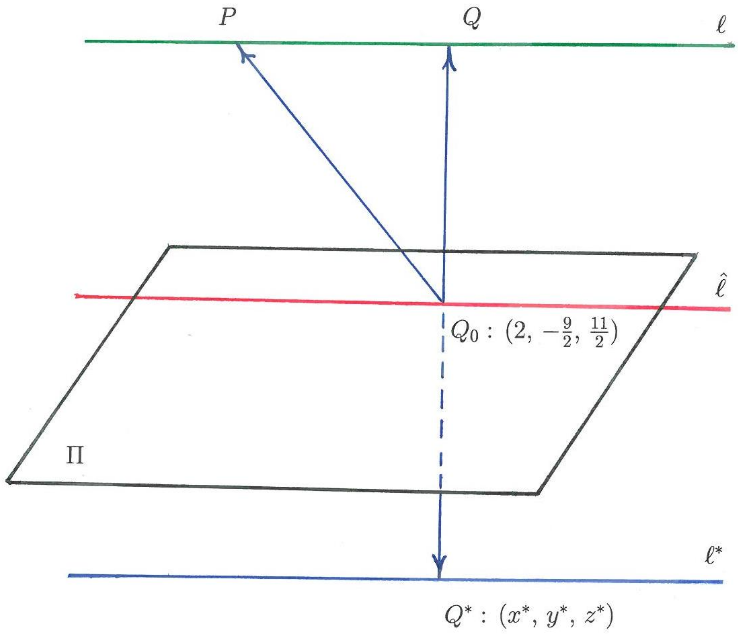

Figure 1.27: The reflection of ℓ about Π

We need to find the coordinates of the point Q^* (see Figure 1.27). Assume that Q^* has the coordinates (x^*, y^*, z^*). By part a) above we know that $Q: \; (3, -4, 6)$ and $Q_0: \; (2, -9/2, 11/2)$. Since $\overrightarrow{Q_0Q^*} = \overrightarrow{QQ_0}$, we have

$$\overrightarrow{Q_0Q^*} = (-1, -\frac{1}{2}, -\frac{1}{2}).$$

But, on the other hand, we have

$$\overrightarrow{Q_0Q^*} = (x^* - 2, y^* + \frac{9}{2}, z^* - \frac{11}{2}),$$

so that

$$\overrightarrow{Q_0Q^*} = (x^* - 2, y^* + \frac{9}{2}, z^* - \frac{11}{2}) = (-1, -\frac{1}{2}, -\frac{1}{2})$$

leads to $x^* = 1,\ y^* = -5$ and $z^* = 5$. Thus we have obtained the coordinates of Q^*, namely

$$Q^* : (1, -5, 5).$$

Now, ℓ^* is passing through the point Q^* and ℓ^* has the same direction vector as ℓ, namely $\mathbf{v} = (1, -5, 3)$. We conclude that the parametric equation of ℓ^* is

$$\ell^* : \begin{cases} x = t + 1 \\ y = -5t - 5 \\ z = 3t + 5 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

Problem 1.5.6.

Consider the plane $\Pi : \; x - y + z = 7$, as well as a triangle with vertices $A : (1, 2, 2)$, $B : (3, 1, 2)$ and $C : (1, 1, 1)$. Note that this triangle is not lying on the plane Π.

a) Orthogonally project the given triangle onto the plane Π and give the vertices of the projected triangle.

b) Reflect the given triangle about the plane Π and give the vertices of the reflected triangle.

Solution 1.5.6.

a) We project the triangle $\triangle ABC$ with vertices $A : (1, 2, 2)$, $B : (3, 1, 2)$ and $C : (1, 1, 1)$ orthogonal onto the plane $\Pi : x - y + z = 7$ and decribe the projected triangle $\triangle A_\Pi B_\Pi C_\Pi$ by calculating the vertices of this triangle, namely the coordinates of A_Π, B_Π and C_Π (see Figure 1.28).

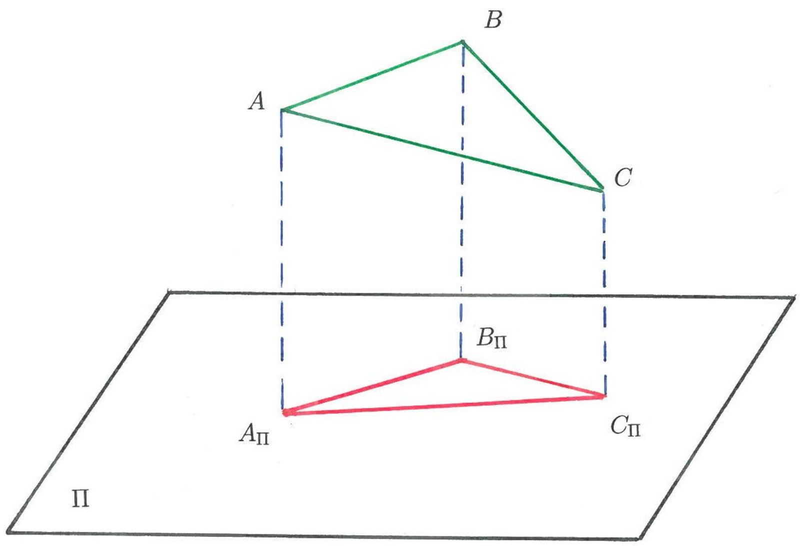

Figure 1.28: $\triangle ABC$ projected orthogonally onto Π resulting in $\triangle A_\Pi B_\Pi C_\Pi$.

Assume that the coordiantes of A_Π, B_Π and C_Π are as follows (see Figure 1.28):

$$A_\Pi : (x_1, y_1, z_1), \quad B_\Pi : (x_2, y_2, z_2), \quad C_\Pi : (x_3, y_3, z_3).$$

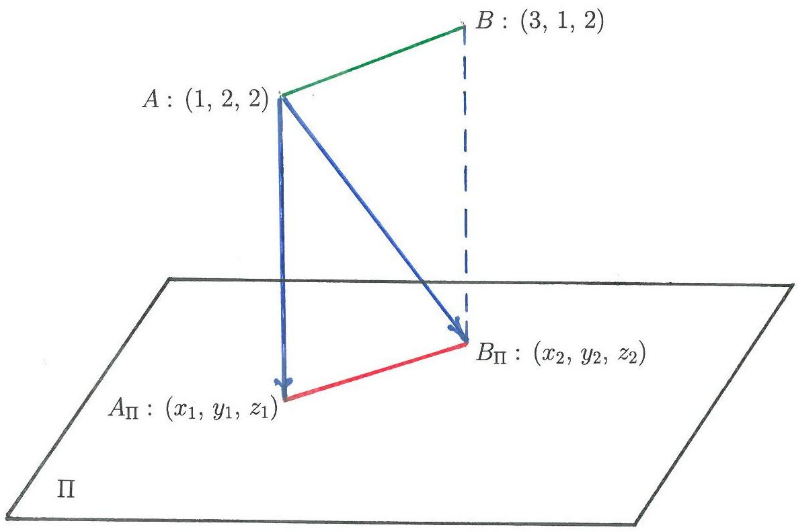

Figure 1.29: Coordinates of A_Π for the projection of $\triangle ABC$

The vector $\overrightarrow{AA_\Pi}$ can be obtained by projecting $\overrightarrow{AB_\Pi}$ orthogonally onto $\mathbf{n}$. That is

$$\overrightarrow{AA_\Pi} = \text{proj}_{\mathbf{n}} \overrightarrow{AB_\Pi} = \left(\frac{\overrightarrow{AB_\Pi} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n},$$

where

$$\overrightarrow{AA_\Pi} = (x_1 - 1, y_1 - 2, z_1 - 2), \quad \overrightarrow{AB_\Pi} = (x_2 - 1, y_2 - 2, z_2 - 2), \quad \mathbf{n} = (1, -1, 1).$$

Calculating we obtain

$$\begin{aligned} \overrightarrow{AA_\Pi} = \text{proj}_{\mathbf{n}} \overrightarrow{AB_\Pi} &= \left(\frac{1(x_2 - 1) - 1(y_2 - 2) + 1(z_2 - 2)}{1 + 1 + 1} \right) (1, -1, 1) \\ &= \left(\frac{x_2 - y_2 + z_2 - 1}{3} \right) (1, -1, 1) = (2, -2, 2), \end{aligned}$$

where $x_2 - y_2 + z_2 = 7$ as this is a point on the plane Π. We have

$$\overrightarrow{AA_\Pi} = (x_1 - 1, y_1 - 2, z_1 - 2) = (2, -2, 2),$$

so that $x_1 = 3,\ y_1 = 0,\ z_1 = 4$ and the coordinates of A_Π are

$$A_\Pi : (3, 0, 4).$$

To find the coordinates of B_Π, we project $\overrightarrow{BA_\Pi}$ orthogonally onto $\mathbf{n}$ (see Figure 1.30), i.e.

$$\overrightarrow{BB_\Pi} = \operatorname{proj}_{\mathbf{n}} \overrightarrow{BA_\Pi} = \left(\frac{\overrightarrow{BA_\Pi} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n},$$

where

$$\overrightarrow{BA_\Pi} = (0, -1, 2), \quad \overrightarrow{BB_\Pi} = (x_2 - 3, y_2 - 1, z_2 - 2), \quad \mathbf{n} = (1, -1, 1).$$

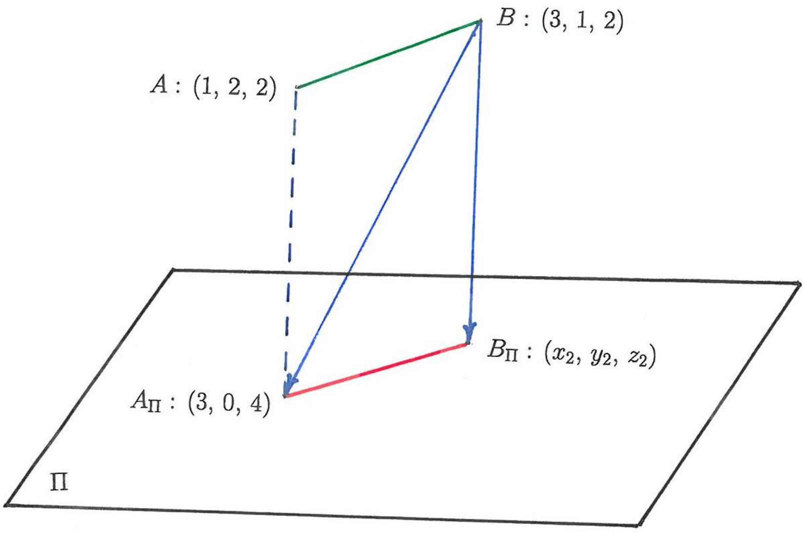

Figure 1.30: Coordinates of B_Π for the projection of $\triangle\, ABC$

Calculating we obtain

$$\overrightarrow{BB_\Pi} = \operatorname{proj}_{\mathbf{n}} \overrightarrow{BA_\Pi} = (1, -1, 1),$$

so that

$$\overrightarrow{BB_\Pi} = (x_2 - 3, y_2 - 1, z_2 - 2) = (1, -1, 1).$$

The coordinates of B_Π are then

$$B_\Pi:\ (4, 0, 3).$$

In the same way, we project $\overrightarrow{CB_\Pi}$ orthogonally onto $\mathbf{n}$ to find C_Π (see Figure 1.31).

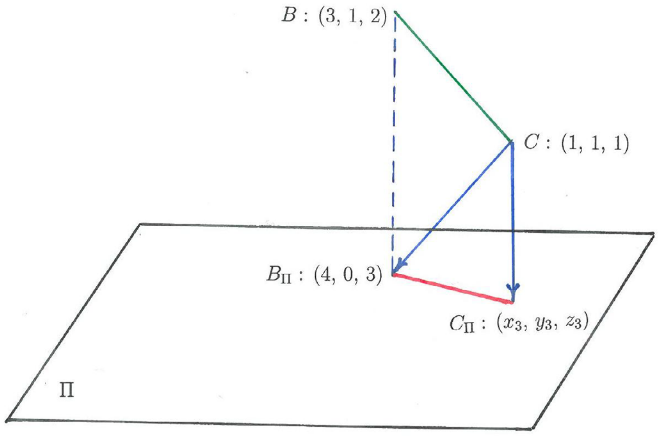

Figure 1.31: Coordinates of C_Π for the projection of $\triangle ABC$

We obtain

$$\overrightarrow{CC_\Pi} = \text{proj}_{\mathbf{n}} \overrightarrow{CB_\Pi} = (2, -2.2)$$

and comparing this with $\overrightarrow{CC_\Pi} = (x_3 - 1, y_3 - 1, z_3 - 1)$, we obtain

$$C_\Pi : (3, -1, 3).$$

This completes the calculations of the vertices for $\triangle A_\Pi B_\Pi C_\Pi$.

b) We reflect the triangle $\triangle ABC$ with vertices $A:\ (1,2,2)$, $B:\ (3,1,2)$ and $C:\ (1,1,1)$ about the plane $\Pi:\ x-y+z=7$ and decribe the reflected triangle $\triangle A^*B^*C^*$ by calculating the vertices of this triangle, namely the coordinates of A^*, B^* and C^* (see Figure 1.32).

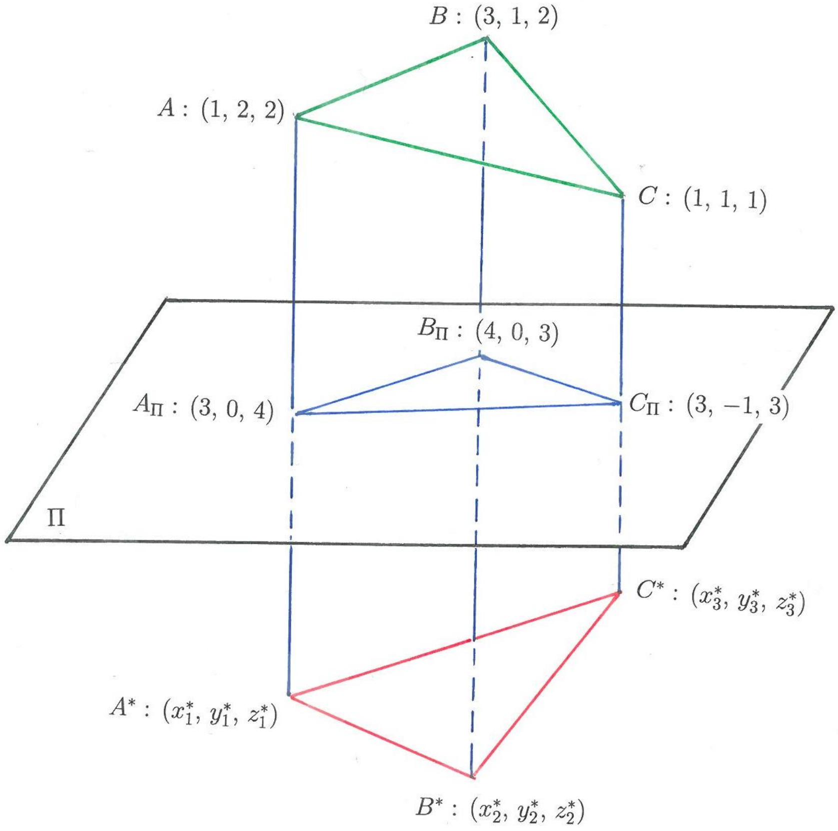

Figure 1.32: $\triangle ABC$ reflected about Π resulting in $\triangle A^*B^*C^*$.

We assume that the coordinates of A^*, B^* and C^* are as follows (see Figure 1.32):

$$A^*:\ (x_1^*, y_1^*, z_1^*),\quad B^*:\ (x_2^*, y_2^*, z_2^*),\quad C^*:\ (x_3^*, y_3^*, z_3^*).$$

From part a) above we have (see also Figure 1.32)

$$\overrightarrow{AA^*} = 2\overrightarrow{AA_\Pi} = 2(2,-2,2) = (4,-4,4),$$

and, moreover,

$$\overrightarrow{AA^*} = (x_1^*-1, y_1^*-2, z_1^*-2) = (4,-4,4).$$

Thus $x_1^* = 5,\ y_1^* = -2$ and $z_1^* = 6$, so we have found the coordiantes of A^*, namely

$$A^* : (5, -2, 6).$$

Also

$$\overrightarrow{BB^*} = 2\overrightarrow{BB_{\text{II}}} = 2(1, -1, 1) = (x_2^* - 3, y_2^* - 1, z_2^* - 2)$$
$$\overrightarrow{CC^*} = 2\overrightarrow{CC_{\text{II}}} = 2(2, -2, 2) = (x_3^* - 1, y_3^* - 1, z_3^* - 1),$$

which leads to the following coordinates for B^* and C^*:

$$B^* : (5, -1, 4), \qquad C^* : (5, -3, 5).$$

1.6 Exercises

1. Consider the following two vectors in $\mathbb{R}^3$: $\mathbf{u} = (-1, 2, 3)$ and $\mathbf{v} = (1, -1, 2)$.

   a) Find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

   [**Answer:** $\text{proj}_{\mathbf{v}}\mathbf{u} = (\frac{1}{2}, -\frac{1}{2}, 1)$.]

   b) Find the orthogonal projection $\mathbf{u}_{yz}$ of $\mathbf{u}$ onto the yz-plane.

   [**Answer:** $\mathbf{u}_{yz} = (0, 2, 3)$.]

   c) Find the vector $\mathbf{u}^*$ that is the reflection of $\mathbf{u}$ about $\mathbf{v}$.

   [**Answer:** $\mathbf{u}^* = (2, -3, -1)$.]

   d) Find the vector $\mathbf{u}^*{}_{xz}$ that is the reflection of $\mathbf{u}$ about the xz-plane.

   [**Answer:** $\mathbf{u}^*{}_{xz} = (-1, -2, 3)$.]

   e) Find the vector that results when $\mathbf{u}$ is first reflected about the xy-plane and then reflected about the yz-plane. Is the resulting vector different if we first reflect about the yz-plane and then reflect about the xy-plane?

   [**Answer:** $(1, 2, -3)$. The vector is the same.]

   f) Find the vector that results when $\mathbf{u}$ is first reflected about the xy-plane and then projected orthogonally onto the yz-plane. Is the resulting vector different if we first project orthogonally onto the yz-plane and then reflect about the xy-plane?

   [**Answer:** $(0, 2, -3)$. The vector is the same.]

2. Find all the values for $a \in \mathbb{R}$, such that the volume of the parallelepiped described by the vectors $\mathbf{u} = (1, 1, 2)$, $\mathbf{v} = (-1, a, 3)$ and $\mathbf{w} = (2, 1, a)$ is one cubic unit.

   [**Answer:** $a \in \{0,\ 1,\ 2,\ 3\}$.]

3. Consider the following three vectors:

$$\mathbf{u} = a(1,1,2), \quad \mathbf{v} = (-1,b,-1), \quad \mathbf{w} = (7,1,c),$$

where a, b and c are real parameters.

a) Find all values for a, b and c, such that the given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ describe a rectangular parallelepiped (i.e. a parallelepiped with perpendicular sides) with a volume of 132 cubic units.

[**Answer:** $a \in \{-2,\, 2\},\ b = 3,\ c = -4.$]

b) Find all values for a, b and c, such that the volume of the parallelepiped, described by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ with $a \neq 0$, is zero cubic units.

[**Answer:** $b = \dfrac{8-c}{c-14}$ for all $c \in \mathbb{R}\backslash\{14\}.$]

4. Consider three points P_1, P_2 and P_3 with the following coordinates in $\mathbb{R}^3$:

$$P_1 :\ (2,-1,1), \quad P_2 :\ (3,2,-1), \quad P_3 :\ (-1,3,2).$$

a) Find the equation of the plane Π_1 that contains the three given points.

[**Answer:** $11x + 5y + 13z = 30.$]

b) Assume that the normal $\mathbf{n}$ of a plane Π_2 is given as $\mathbf{n} = (-2,1,4)$ and that Π_2 contains the given point P_1. Find the equation of Π_2.

[**Answer:** $-2x + y + 4z = -1.$]

c) Find the angle θ between the two planes Π_1 and Π_2 that you have obtained in part a) and part b).

[**Answer:** $\theta = \arccos\left(\dfrac{\sqrt{15}}{9}\right).$]

5. Consider a line ℓ in $\mathbb{R}^3$ that passes through the points $P_1 :\ (1,-2,-1)$ and $P_2 :\ (3,-1,1)$.

a) Find a parametric equation for ℓ.

[**Answer:**

$$\ell : \begin{cases} x = 2t + 1 \\ y = t - 2 \\ z = 2t - 1 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$]

b) Find the distance s from the origin $(0, 0, 0)$ to the line ℓ that you have obtained in part a).

[**Answer:** $s = \dfrac{5\sqrt{2}}{3}$.]

6. Consider a triangle with vertices $A :\ (1, 0, 1)$, $B :\ (2, 1, -1)$ and $C :\ (2, 2, 1)$.

a) Find the distance from the point B to the base of the triangle with vertices A and C.

[**Answer:** $\sqrt{\dfrac{21}{5}}$.]

b) Find the area of the triangle ABC by making use of the cross product.

[**Answer:** $\dfrac{1}{2}\sqrt{21}$.]

7. Consider the following two lines in $\mathbb{R}^3$:

$$\ell_1 : \begin{cases} x = 2t + 3 \\ y = -4t + 1 \\ z = 2t + 2 \quad \text{for all } t \in \mathbb{R}, \end{cases} \qquad \ell_2 : \begin{cases} x = -s \\ y = s + 3 \\ z = -s - 1 \quad \text{for all } s \in \mathbb{R}. \end{cases}$$

Do the lines intersect? If so, find the point of intersection for this case.

[**Answer:** The point of intersection is $(4, -1, 3)$.]

8. Consider a pyramid $ABCD$ with vertices at $A:\ (2,1,0)$, $B:\ (0,2,3)$, $C:\ (1,0,1)$ and $D:\ (1,1,1)$ as shown in Figure 1.33.

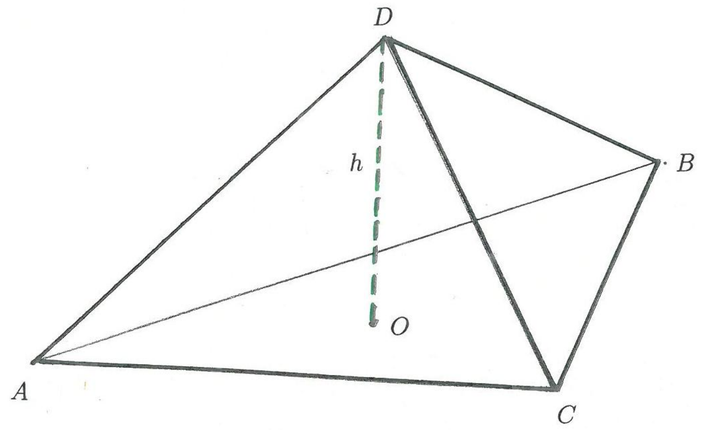

Figure 1.33: The pyramid $ABCD$.

Find the height of this pyramid.

[**Answer:** The height of the pyramid is given by the distance from the point D to the plane that contains the triangle ABC, namely $\dfrac{1}{\sqrt{26}}$.]

9. Consider the line ℓ, given in parametric form by

$$\ell : \begin{cases} x = kt + 2 \\ y = t - 3 \\ z = 3t + 4 \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

and the plane Π, given by the equation

$$\Pi : \; 3x + 2y + 4z = 1.$$

a) Determine for which value(s) of $k \in \mathbb{R}$, if any, is ℓ parallel to Π.

[**Answer:** $k = -\dfrac{14}{3}$.]

b) Find the distance from ℓ to Π for those values of k for which ℓ is parallel to Π, if any such values exist.

[**Answer:** $\dfrac{15}{\sqrt{29}}$.]

c) Find the intersection of ℓ with Π, for all those values of k for which ℓ is not parallel to Π, if any such values exist.

[**Answer:** The coordinates of intersection is $\dfrac{1}{3k+14}(-9k+28,\ -9k-57,\ 12k+11)$ for all $k \in \mathbb{R}\backslash\{-\dfrac{14}{3}\}$.]

10. Consider the line ℓ given in parametric form by

$$\ell : \begin{cases} x = -t + 2 \\ y = 3t \\ z = 5t - 1 \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

Find all points on ℓ, for which the distance from those points to the plane $x+y-z = 2$ is $\dfrac{2}{\sqrt{3}}$ units.

[**Answer:** The points with coordinates $(\dfrac{7}{3}, -1, -\dfrac{8}{3})$ and $(1, 3, 4)$.]

11. Consider the plane $\Pi: \ x - 2y + 3z = 31$ and the line ℓ given in parametric form by

$$\ell : \begin{cases} x = -t + 2 \\ y = t - 1 \\ z = -2t + 3 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

a) Find the parametric equation of the line $\hat{\ell}$, such that $\hat{\ell}$ is the orthogonal projection of ℓ onto Π.

[**Answer:**

$$\hat{\ell} : \begin{cases} x = -5t + 4 \\ y = -4t - 3 \\ z = -t + 7 \qquad \text{for all } t \in \mathbb{R}. \ ] \end{cases}$$

b) Find the parametric equation of the line ℓ^*, such that ℓ^* is the reflection of ℓ about Π.

[**Answer:**

$$\ell^* : \begin{cases} x = 2t + 4 \\ y = -11t - 3 \\ z = 7t + 13 \qquad \text{for all } t \in \mathbb{R}. \ ] \end{cases}$$

c) Find all points on ℓ, for which the shortest distance between those points and the plane Π is $3/\sqrt{14}$.

[**Answer:** The points $(\frac{11}{3}, -\frac{8}{3}, \frac{19}{3})$ and $(\frac{13}{3}, -\frac{10}{3}, \frac{23}{3})$.]

12. Consider the plane $\Pi: \ 3x + 4y - 5z = 11$ and the line ℓ given in parametric form by

$$\ell : \begin{cases} x = t + 4 \\ y = -2t + 1 \\ z = -t + 3 \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

where ℓ is parallel to Π. Find the line $\hat{\ell}$, such that $\hat{\ell}$ is the orthogonal projection of ℓ onto Π.

[**Answer:**

$$\hat{\ell} : \begin{cases} x = t + \dfrac{28}{5} \\ y = -2t - \dfrac{1}{5} \\ z = -t + 1 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

]

13. Consider the plane $\Pi : \; 2x - y + 2z = -5$ and the triangle $\triangle ABC$ with vertices $A : (1, 0, 1)$, $B : (0, 1, 1)$ and $C : (1, 1, 0)$.

 a) Find the vertices of the triangle $\triangle A_\Pi B_\Pi C_\Pi$, such that $\triangle A_\Pi B_\Pi C_\Pi$ is the orthogonal projection of $\triangle ABC$ onto Π.

 [**Answer:** The vertices of the projected triangle $\triangle A_\Pi B_\Pi C_\Pi$ are $A_\Pi : (-1, 1, -1)$, $B_\Pi : (-\frac{4}{3}, \frac{5}{3}, -\frac{1}{3})$ and $C_\Pi : (-\frac{1}{3}, \frac{5}{3}, -\frac{4}{3})$.]

 b) Find the vertices of the triangle $\triangle A^* B^* C^*$, such that $\triangle A^* B^* C^*$ is the reflection of $\triangle ABC$ about Π.

 [**Answer:** The vertices of the reflection triangle $\triangle A^* B^* C^*$ are $A^* : (-3, 2, -3)$, $B^* : (-\frac{8}{3}, \frac{7}{3}, -\frac{5}{3})$ and $C^* : (-\frac{5}{3}, \frac{7}{3}, -\frac{8}{3})$.]

14. Show that the distance s between two parallel planes,

$$\Pi_1 : \; ax + by + cz = d_1$$

$$\Pi_2 : \; ax + by + cz = d_2,$$

is given by

$$s = \frac{|d_1 - d_2|}{\|\mathbf{n}\|},$$

where $\mathbf{n} = (a, b, c)$ is the normal vector for the planes (see Theoretical Remark 1.5).

Chapter 2

Matrix algebra and Gauss elimination

The aim of this chapter:

We introduce points in the Euclidean space $\mathbb{R}^n$ in terms of n-component vectors. Those vectors can be represented in terms of column-matrices (or row-matrices). Any system of linear equations can in fact be written in the form of a matrix equation, namely $A\mathbf{x} = \mathbf{b}$, which can subsequently be investigated using matrix properties. To achieve this, we introduce *addition* and *multiplication* of matrices, the *determinant* of a square matrix and the *inverse* of a square matrix (for those matrices that are invertible). For solving systems of linear equations, we use the method of *Gauss elimination* and also introduce an alternate method following *Cramer's rule*, by which certain types of square linear systems can be solved.

2.1 Matrix operations of addition and multiplication

We introduce vectors in the Euclidean space $\mathbb{R}^n$ and describe the basic vector operations.

Theoretical Remarks 2.1.

1. **Vectors in** $\mathbb{R}^n$:

   A vector $\mathbf{u}$ in the Euclidean space $\mathbb{R}^n$ is an n-tuple $(u_1, u_2, \ldots, u_n)$. We write

   $$\mathbf{u} = (u_1, u_2, \ldots, u_n).$$

   Here $u_1,\ u_2,\ \ldots, u_n$ are numbers (real or complex, although we consider only real numbers in this book). Every n-tuple denotes a unique vector or point in $\mathbb{R}^n$. We can represent $\mathbf{u}$ by an $n \times 1$ column-matrix

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

or we can represent $\mathbf{u}$ by an $1 \times n$ row-matrix

$$\mathbf{u} = (u_1 \; u_2 \; \ldots \; u_n).$$

Consider, furthermore, the vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. We have the following

Properties:

- $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + u_2, \ldots, u_n + v_n) = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $r\,\mathbf{u} = (ru_1, ru_2, \ldots, ru_n) = \mathbf{u}\,r$ for all $r \in \mathbb{R}$.
- $0\mathbf{u} = \mathbf{0} = (0, 0, \ldots, 0)$ called the **zero-vector** of $\mathbb{R}^n$.

2. **Matrix addition and multiplication with constants:**

Consider the following $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where a_{ij} are numbers (real or complex, although we consider only real numbers in this book).

Note: *In some cases it is convenient to denote matrix A as follows:*

$$A = [a_{ij}] \quad \text{or} \quad A = [\mathbf{a_1} \; \mathbf{a_2} \; \ldots \; \mathbf{a_n}], \quad \text{where } \mathbf{a_j} \in \mathbb{R}^m.$$

Consider two matrices of size $m \times n$, namely

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}].$$

We define addition and multiplications as follows:

Addition of matrices:

$$A + B = [a_{ij} + b_{ij}].$$

Multiplication with a constant:

$$rA = [ra_{ij}] = Ar \quad \text{for all } r \in \mathbb{R}$$

$$0\,A = [0\,a_{ij}] = 0_{mn}, \quad \text{where } 0_{mn} \text{ denotes the } m \times n \text{ zero matrix.}$$

We have the following

Properties:

Let A, B and C be matrices of size $m \times n$ and let r and s be any real numbers. Then

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0_{mn} = A$
- $r(A + B) = rA + rB$
- $(r + s)A = rA + sA$
- $r(sA) = (rs)A.$

3. **Matrix-vector multiplication:**

Consider the $m \times n$ matrix A, namely

$$A = [\mathbf{a_1}\ \mathbf{a_2}\ \ldots\ \mathbf{a_n}], \quad \mathbf{a_j} \in \mathbb{R}^m,$$

and consider the vector $\mathbf{x} \in \mathbb{R}^n$, namely

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The matrix-vector product $A\mathbf{x}$ is a vector in $\mathbb{R}^m$ defined as follows:

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}.$$

Let A be an $m \times n$ matrix, let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$ and let r be any real number. We have the following

Properties:

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $rA(\mathbf{u}) = A(r\mathbf{u})$.

4. **Matrix-matrix multiplication:**

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix, where

$$B = [\mathbf{b_1}\ \mathbf{b_2}\ \cdots\ \mathbf{b_p}], \quad \mathbf{b_j} \in \mathbb{R}^n.$$

The matrix-matrix product AB is a matrix of size $m \times p$ defined as follows:

$$AB = [A\mathbf{b_1}\ A\mathbf{b_2}\ \cdots A\mathbf{b_p}].$$

For the properties listed below, we assume that the matrices A, B and C are of the correct size, such that the listed properties do not contradict the above given definitions. Let r be any real number. We have the following

Properties:

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$

- $(A+B)C = AC + BC$
- $r(AB) = (rA)B = A(rB)$.

The $n \times n$ **identity matrix**, denoted by I_n, is defined as follows:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = [\mathbf{e_1}\ \mathbf{e_2}\ \cdots \mathbf{e_n}],$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \cdots,\ \mathbf{e_n}\}$ is the standard basis for $\mathbb{R}^n$, namely

$$\mathbf{e_1} = (1, 0, \ldots, 0), \quad \mathbf{e_2} = (0, 1, \ldots, 0), \quad \ldots, \quad \mathbf{e_n} = (0, 0, \ldots, 1).$$

Let A be an $m \times n$ matrix and let $\mathbf{u} \in \mathbb{R}^n$. Then

- $AI_n = A = I_m A$.
- $I_n \mathbf{u} = \mathbf{u}$.
- $I_n^p = I_n$ for all $p \in \mathbb{N}$.

Remark: *Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then the product*

$$AB = 0_{mm},$$

where 0_{mm} denotes the $m \times m$ zero matrix, does **not** *imply that A is a zero matrix or that B is a zero matrix. For example,*

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 2.1.1.

Consider the following three matrices:

$$A = \begin{pmatrix} a & 2 & 1 \\ 1 & 0 & a \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & a & 0 \\ 0 & 1 & 2a^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

where a is an unspecified real parameter. Find all values for a, such that

$$A + B = C.$$

Solution 2.1.1.

We add matrices A and B and compare every entry of the resulting matrix with the corresponding entries in matrix C:

$$\begin{pmatrix} a+2 & 2+a & 1 \\ 1 & 1 & a+2a^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We obtain

$$a+2=1, \quad 2+a=1, \quad a+2a^2=1,$$

for which $a=-1$ is the only common solution.

Problem 2.1.2.

Consider the following two matrices:

$$A = \begin{pmatrix} a & b \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

where a and b are unspecified real parameters. Find all values for a and b, such that

$$AB = BA.$$

Solution 2.1.2.

We multiply the matrices A and B in the order AB:

$$AB = \begin{pmatrix} a & b \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ 3 & 2 \end{pmatrix}.$$

For the multiplication BA, we obtain

$$BA = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} a+2 & b+4 \\ a & b \end{pmatrix}.$$

Comparing now every entry in AB with the corresponding entries in BA, we obtain

$$a+b=a+2, \quad 2a=b+4, \quad a=3, \quad b=2.$$

The above system of equations has the solution $a=3$ and $b=2$. Thus for the values $a=2$ and $b=3$ in A, the matrices A and B commute, i.e. $AB=BA$, and for all other values of a and b, the matrix multiplication does not commute, i.e. $AB \neq BA$.

Problem 2.1.3.

Consider the matrix

$$A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

where b and c are unspecified real parameters. Find all values of b and c, such that

$$A^2 = I_2,$$

where I_2 is the 2×2 identity matrix.

Solution 2.1.3.

We calculate A^2:

$$A^2 = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix}.$$

The 2×2 identity matrix I_2 is

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Comparing each entry in A^2 with the corresponding entries of I_2, we obtain

$$bc = 1.$$

We conclude that the matrix

$$A = \begin{pmatrix} 0 & b \\ \frac{1}{b} & 0 \end{pmatrix}.$$

satisfies the relation $A^2 = I_2$ for all $b \in \mathbb{R} \backslash \{0\}$.

2.2 The determinant of square matrices

We introduce the determinant of square matrices and show how to compute those using the cofactor expansion and elementary row operations.

Theoretical Remarks 2.2.

1. The **determinant** of an $n \times n$ matrix $A = [a_{ij}]$, denoted by $\det A$ or $|A|$, is a number that can be calculated by the **cofactor expansion** across the i^{th} row,

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

or, alternately, $\det A$ can be calculated by the cofactor expansion down the j^{th} column,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Here the number C_{ij} is the (i, j)-**cofactor** of A, namely

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

where A_{ij} denotes the $(n-1) \times (n-1)$ matrix, obtained from matrix A by removing the i^{th} row and the j^{th} column in A.

2. Two matrices A and B are said to be **row equivalent** (we write $A \sim B$) if B can be obtained from A by applying a finite number of elementary row operations on A. The three **elementary row operations** are the following:

   i. Replace one row by adding that row to the multiple of another row.
   ii. Interchange two rows.
   iii. Multiply all entries in a row by a nonzero constant k.

3. The calculations for the determinant of A can be simplified by applying elementary row operations on A. The relation between the determinant of A and the determinant of its row equivalent matrices, are as follows:

   - If $A \sim B$, where B was obtained by applying the elementary row operation (i) on A, then $\det B = \det A$.
   - If $A \sim B$, where B was obtained by applying the elementary row operation (ii) on A, then $\det B = -\det A$.
   - If $A \sim B$, where B was obtained by applying the elementary row operation (iii) on A, then $\det B = k \det A$.

4. Let A and B be $n \times n$ matrices. Then we have the following

Properties:

- $\det(AB) = (\det A)(\det B)$
- $\det(A^m) = (\det A)^m$ for any $m \in \mathbb{N}$.

- $\det A^T = \det A$.
  Note: *The* **transpose** *of any $m \times n$ matrix B is an $n \times m$ matrix B^T, where the columns in B are the rows in B^T.*
- $\det(cA) = c^n \det A$ for any number c.
- The determinant of a diagonal matrix is given by the product of all its diagonal elements.
- The determinant of a lower triangular matrix or an upper triangular matrix is given by the product of all its diagonal elements.
  Note: *A square matrix is said to be* **lower triangular** *if all the entries above its diagonal enties are zero elements. Similarly, a square matrix is said to be* **upper triangular** *if all the entries below its diagonal entries are zero elements.*
- $\det I_n = 1$, where I_n is the $n \times n$ identity matrix.
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$, where A^{-1} denotes the **inverse** of the matrix A.

  Note: *For details on the inverse of matrices and how to obtain the inverse, see* **Theoretical Remark 2.3**.

5. Consider three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, in $\mathbb{R}^3$. Then

- the area of the parallelogram described by $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is given by the norm of the cross product

$$\|\mathbf{u} \times \mathbf{v}\| = \| \det \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \|;$$

- the volume of the parallelepiped described by $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ is given by the triple product

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = | \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} |.$$

  See **Theoretical Remark 1.2** for details regarding the cross product and the triple product for vectors in $\mathbb{R}^3$.

Problem 2.2.1.

Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 1 \\ -1 & -2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & -1 \\ 3 & 0 & 1 & 2 \\ -1 & 4 & 1 & 2 \end{pmatrix}.$$

Solution 2.2.1.

Below we compute the determinant of the given matices by cofactor expansions across the 1st row.

$$\det A = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} = a_{11}(-1)^{1+1}\det A_{11} + a_{12}(-1)^{1+2}\det A_{12}$$
$$= 3 - 8 = -5.$$

$$\det B = \begin{vmatrix} 1 & -2 & 2 \\ 3 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}$$
$$= 1(1+2) + 2(3+1) + 2(-6+1) = 1.$$

$$\det C = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & -1 \\ 3 & 0 & 1 & 2 \\ -1 & 4 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 4 & 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ -1 & 4 & 2 \end{vmatrix}$$
$$= 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}$$
$$- \left(2\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} 2\begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix}\right)$$
$$= -12$$

It is less tedious to compute the determinant by finding the row equivalent upper triangular matrix. We now use this procedure and again compute $\det C$:

$$\det C = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & 3 & 1 & -1 \\ 0 & 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$
$$= (1)(3)(-1)(4) = -12.$$

Problem 2.2.2.

Consider matrix C in problem a) and compute the following:

$$\det(C^4), \quad (\det C)^4, \quad \det(3C), \quad \det(C^T).$$

Solution 2.2.2.

$\det(C^4) = (\det C)^4 = (-12)^4 = 20736,$

$(\det C)^4 = (-12)^4 = 20736,$

$\det(3C) = 3^4 \det C = (81)(-12) = -972,$

$\det(C^T) = \det C = -12$

Problem 2.2.3.

Consider the matrix

$$A = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

Find matrix A^p for all $p \in \mathbb{N}$ by calculating A^2, $A^3, \ldots$.

Solution 2.2.3.

We calculate A^2 and A^3:

$$A^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$A^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for all natural numbers} \quad p \geq 3.$$

Problem 2.2.4.

Consider

$$X^2 - 2X + I_2 = 0_{22}, \tag{2.2.1}$$

where X is a 2×2 matrix, I_2 is the 2×2 identity matrix and 0_{22} is the 2×2 zero matrix .

a) Show that

$$X = \begin{pmatrix} 2 & 1/2 \\ -2 & 0 \end{pmatrix}.$$

is a solution of (2.2.1) and find another solution by factorizing the matrix equation (2.2.1)

b) Show that $\det X = 1$ for a solution X of (2.2.1), even in the case where X is an $n \times n$ matrix.

c) Show that (2.2.1) admits in fact infinitely many solutions.

Solution 2.2.4.

a) We calculate

$$\begin{pmatrix} 2 & 1/2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1/2 \\ -2 & 0 \end{pmatrix} - 2\begin{pmatrix} 2 & 1/2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and obtain the zero matrix. The matrix equation (2.2.1) can be factorized as follows:

$$(X - I_2)^2 = 0_{22}, \quad \text{so that} \quad X = I_2 \quad \text{is another solution for (2.2.1).} \tag{2.2.2}$$

b) It should be clear that the factorization and solution given in (2.2.2) is true when X is an $n \times n$ matrix for any n, so that $X = I_n$ is a solution. Then $\det X = \det I_n = 1$.

c) We let

$$X - I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and calculate} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This leads to the following four conditions:

$$a^2 + bc = 0, \quad b(a+d) = 0, \quad c(a+d) = 0, \quad cb + d^2 = 0. \tag{2.2.3}$$

By subtracting the first equation from the fourth equation above, we obtain the equivalent system

$$a^2 + bc = 0, \quad b(a+d) = 0, \quad c(a+d) = 0, \quad (a+d)(a-d) = 0.$$

Investigating the two cases $a + d = 0$ and $a - d = 0$ we come to the conclusion that $a + d = 0$ and $a^2 + bc = 0$ is the only case that provides all solutions. Thus

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & 1-a \end{pmatrix}$$

for all $a,\ b$ and c such that $a^2 + bc = 0$. Moreover, $\det X = 1$. The matrix equation

$$X^2 - 2X + I_2 = 0_{22}$$

has therefore infinitely many solutions.

2.3 The inverse of square matrices

In this section we introduce the inverse of a square matrix and show how to find this inverse, for invertible square matrices. The determinant of the matrices plays a central role in this discussion.

Theoretical Remarks 2.3.

Let A be an $n \times n$ matrix. The matrix A is said to be **invertible** if there exists another $n \times n$ matrix A^{-1}, called the **inverse** of A, such that

$$A^{-1}A = AA^{-1} = I_n.$$

A square matrix that is not invertible is said to be **singular**.

We have the following statements:

1. Matrix A is invertible if and only if $\det A \neq 0$.

2. Matrix A is invertible if and only if A is row equivalent to the $n \times n$ identity matrix I_n. That is, A is invertible if and only if the reduced echelon form of $[A\ I_n]$ is $[I_n\ A^{-1}]$.

3. If matrix A is invertible, then

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det A},$$

where $\operatorname{adj}(A)$ denotes the **adjugate** of A, given by the matrix

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots\vdots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

Here C_{ij} is the (i, j)-cofactor of A, namely

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Let A and B be $n \times n$ invertible matrices. Then we have the following

Properties:

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = c^{-1}A^{-1}$ for nonzero numbers c.
- $(A^T)^{-1} = (A^{-1})^T$, where A^T is the **transpose** of A.
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\det(A^{-1}) = \dfrac{1}{\det A}$.

Problem 2.3.1.

Consider three $n \times n$ matrices, namely A, B and C, where A and B are invertible matrices such that the following matrix equation is satisfied:

$$A^2B + A = AC.$$

Find A.

Solution 2.3.1.

We multiply the given matrix equation

$$A^2B + A = AC$$

by A^{-1} from the left, i.e.

$$A^{-1}A^2B + A^{-1}A = A^{-1}AC,$$

and obtain

$$AB + I_n = C.$$

We now multiply the previous matrix equation by B^{-1} from the right, i.e.

$$ABB^{-1} + I_nB^{-1} = CB^{-1},$$

where $I_nB^{-1} = B^{-1}$ and $ABB^{-1} = AI_n = A$. Thus

$$A = CB^{-1} - B^{-1}.$$

Problem 2.3.2.

Let X, A and B be $n \times n$ matrices, where A and X are invertible matrices such that

$$BX^{-1} + 2A = BAX^{-1}.$$

a) Find X.

b) Find X, such that

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Solution 2.3.2.

a) We multiply the given matrix equation

$$BX^{-1} + 2A = BAX^{-1}$$

by X from the right to obtain

$$B + 2AX = BA \quad \text{or} \quad AX = \frac{1}{2}BA - \frac{1}{2}B.$$

We now multiply the provious matrix equation by A^{-1} from the left to obtain

$$X = A^{-1}\left(\frac{1}{2}BA - \frac{1}{2}B\right) \quad \text{or} \quad X = \frac{1}{2}A^{-1}B\left(A - I_n\right).$$

b) For

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

we have

$$A - I_2 = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{pmatrix}.$$

Inserting this into the result that was obtained for X above, namely $X = \frac{1}{2}A^{-1}B(A - I_2)$, we obtain

$$X = \begin{pmatrix} 3/4 & 1 \\ 9/8 & 1/4 \end{pmatrix}.$$

Problem 2.3.3.

Calculate the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Solution 2.3.3.

To gain the inverse of matrix A, we find the reduced echelon form of $[A\ I_4]$. We obatin

$$[A\ I_4] = \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1 \\ 0 & 1 & 1 & 0 & -1/2 & 3/2 & 1/2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & -1 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & 1 \end{array}\right).$$

We conclude that the inverse of A is

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1/2 & 1/2 & -1/2 & 0 \\ 0 & 1 & 1 & -1 \\ 1/2 & -1/2 & -1/2 & 1 \end{pmatrix}.$$

Problem 2.3.4.

Find all real values of k such that the given matrix A is invertible and calculate then the inverse of the matrix for one of those values of k:

$$A = \begin{pmatrix} k & 1 & 2 \\ 2 & 1 & k \\ k & 0 & 1 \end{pmatrix}.$$

Solution 2.3.4.

First we find all real values of k for which A is singular. That is, we find k such that

$$\det A = 0, \quad \text{where} \quad \begin{vmatrix} k & 1 & 2 \\ 2 & 1 & k \\ k & 0 & 1 \end{vmatrix} = k^2 - k - 2 \quad \text{so that} \quad (k+1)(k-2) = 0.$$

Hence A is singular for $k = -1$ or $k = 2$, and therefore A is invertible for all $k \in \mathbb{R} \backslash \{-1,\ 2\}$. We choose the value $k = 0$ and calculate the inverse of A:

$$[A\ I_3] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

The inverse of A is thus

$$A^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.4 Gauss elimination for systems of linear equations

In this section we describe the method of Gauss elimination to solve systems of linear equations. We prove that any consistent linear system admits either a unique solution or it admits infinitely many solutions (see Problem 2.4 c below)

Theoretical Remarks 2.4.

A system of m **linear equations** in n unknown variables $x_1,\ x_2,\ \ldots,\ x_n$ can be written in the form of a matrix equation

$$A\,\mathbf{x} = \mathbf{b}, \tag{2.4.1}$$

where A is a given $m \times n$ matrix, called the coefficient matrix of the system, $\mathbf{b}$ is a given vector in $\mathbb{R}^m$ and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

1. The matrix equation (2.4.1) is said to be **consistent**, if there exists at least one solution $\mathbf{x} \in \mathbb{R}^n$ which satisfies (2.4.1). The matrix equation (2.4.1) is said to be **inconsistent** (or **incompatible**), if there exists no $\mathbf{x} \in \mathbb{R}^n$ that satisfies (2.4.1).

2. Any consistent matrix equation of the form (2.4.1) admits either a **unique solution** $\mathbf{x} \in \mathbb{R}^n$, or it admits **infinitely many solutions** $\mathbf{x} \in \mathbb{R}^n$.

3. All solutions $\mathbf{x} \in \mathbb{R}^n$ of (2.4.1) can be obtained by the so-called **Gauss elimination method**, which can be described by the following four steps:

Step I. Write down the **augmented matrix** $[A\ \mathbf{b}]$ of (2.4.1).

Step II. Apply **elementary row operations** on $[A\ \mathbf{b}]$ to convert $[A\ \mathbf{b}]$ into **row equivalent matrices**.

Step III. Apply Step II until $[A\ \mathbf{b}]$ is in its unique **reduced echelon form**, which we denote by $[B\ \mathbf{c}]$. The matrix equation

$$B\mathbf{x} = \mathbf{c}, \quad \mathbf{c} \in \mathbb{R}^m \tag{2.4.2}$$

is then the simplest form of the original system $A\mathbf{x} = \mathbf{b}$. System (2.4.2) has the same solutions as system (2.4.1).

Step IV. Solve equation (2.4.2). The columns in matrix B with the leading $\mathbf{1}$'s are the so-called **pivot columns** of matrix A. Every column j in the coefficient matrix that is not a pivot column implies that x_j is an arbitrary parameter in the solution $\mathbf{x} = (x_1, x_2, \ldots, x_j, \ldots, x_n)$ of (2.4.1). If the last column in the matrix $[B\ \mathbf{c}]$ is a pivot column, then system (2.4.1) is inconsistent.

Problem 2.4.1.

Find all solutions $\mathbf{x} \in \mathbb{R}^5$ of the system of linear equations given by the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 3 \\ 2 & 0 & 1 & 3 & 9 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ -5 \end{pmatrix}.$$

Solution 2.4.1.

The augmented matrix $[A\ \mathbf{b}]$ of the given system is

$$[A\ \mathbf{b}] = \begin{pmatrix} 0 & 1 & 2 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 3 & -4 \\ 2 & 0 & 1 & 3 & 9 & -5 \end{pmatrix}.$$

Applying elementary row operations, we bring $[A\ \mathbf{b}]$ in its unique reduced echelon form:

$$[A\ \mathbf{b}] \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 2 & 0 & 1 & 3 & 9 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -3 & 0 & 10 & 10 & 0 \\ 0 & 1 & 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & 4 & -3 \\ 0 & 1 & 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ the simplified, but equivalent, linear system then takes the form

$$\begin{aligned} x_1 + x_4 + 4x_5 &= -3 \\ x_2 - 3x_4 - 2x_5 &= -1 \\ x_3 + x_4 + x_5 &= 1. \end{aligned}$$

From the above reduced echelon form of $[A\ \mathbf{b}]$, we conclude that the 4^{th} and 5^{th} columns in the coefficient matrix A are not pivot columns. It therefore follows that x_4 and x_5 are free parameters in the solutions. We let say $x_4 = t$ and $x_5 = s$, so that

$$x_1 = -t - 4s - 3, \quad x_2 = 3t + 2s - 1, \quad x_3 = -t - s + 1.$$

All the solutions of the given linear system $A\mathbf{x} = \mathbf{b}$ are

$$\mathbf{x} = \begin{pmatrix} -t - 4s - 3 \\ 3t + 2s - 1 \\ -t - s + 1 \\ t \\ s \end{pmatrix} \quad \text{for all} \quad t \in \mathbb{R} \text{ and all } s \in \mathbb{R}.$$

The solutions can also be presented in the following form:

$$\mathbf{x} = t\begin{pmatrix} -1 \\ 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } \ t \in \mathbb{R} \text{ and all } s \in \mathbb{R}.$$

Problem 2.4.2.

Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 2 & k & 2 \\ 3 & k & 18 & 6 \\ 1 & 1 & 6 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that the given system is consistent and give then all solutions of this consistent system.

b) Do there exist values of k, such that the given system has a unique solution?

Solution 2.4.2.

a) The augmented matrix for the given system is

$$[A\ \mathbf{b}] = \begin{pmatrix} 1 & 2 & k & 2 & 1 \\ 3 & k & 18 & 6 & 5 \\ 1 & 1 & 6 & 2 & 1 \end{pmatrix}$$

Performing elementary row operations on this augmented matrix, we obtain the following row equivalent matrices for $[A\ \mathbf{b}]$:

$$[A\ \mathbf{b}] \sim \begin{pmatrix} 1 & 2 & k & 2 & 1 \\ 0 & k-6 & -3(k-6) & 0 & 2 \\ 0 & 1 & k-6 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & k & 2 & 1 \\ 0 & 1 & (k-6) & 0 & 0 \\ 0 & k-6 & -3(k-6) & 0 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & k & 2 & 1 \\ 0 & 1 & (k-6) & 0 & 0 \\ 0 & 0 & -(k-6)(k-3) & 0 & 2 \end{pmatrix}.$$

From the last echelon form of $[A\ \mathbf{b}]$ above, it is clear that the given system $A\mathbf{x} = \mathbf{b}$ is consistent for all $k \in \mathbb{R}\backslash\{3,\ 6\}$. Therefore, the system is inconsistent if and only if $k = 3$ or $k = 6$. We now solve the system for those values of k for which it is consistent. From the last echelon form above, we have the following simplified system of equations for $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} = (x_1, x_2, x_3, x_4)$:

$$\begin{aligned} &x_1 + 2x_2 + kx_3 + 2x_4 = 1 \\ &x_2 + (k-6)x_3 = 0 \\ &-(k-6)(k-3)x_3 = 2. \end{aligned}$$

Since the 4^{th} column of A is not a pivot column, we know that x_4 can be chosen as a free parameter, say $x_4 = t$. We obtain

$$x_1 = \frac{2(12-k)}{(k-6)(k-3)} - 2t + 1, \quad x_2 = \frac{2}{k-3}, \quad x_3 = -\frac{2}{(k-6)(k-3)},$$

so that the solutions of the given system are

$$\mathbf{x} = t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{(k-6)(k-3)}\begin{pmatrix} k^2 - 11k + 42 \\ 2(k-6) \\ -2 \\ 0 \end{pmatrix} \text{ for all } t \in \mathbb{R} \text{ and all } k \in \mathbb{R}\backslash\{3,\ 6\}.$$

b) It is clear from the last echelon form of $[A\ \mathbf{b}]$ given in part a) above, that the 4^{th} column of A is not a pivot column, and this is always the case for any choice of k. Thus the given system $A\mathbf{x} = \mathbf{b}$ cannot have a unique solution, for any choice of k.

Problem 2.4.3.

Prove that any consistent system, $A\mathbf{x} = \mathbf{b}$ with A an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, will either admit exactly one solution or infinitely many solutions $\mathbf{x} \in \mathbb{R}^n$.

Solution 2.4.3.

Assume that $\mathbf{x_1} \in \mathbb{R}^n$ and $\mathbf{x_2} \in \mathbb{R}^n$ are two distinct solutions for $A\mathbf{x} = \mathbf{b}$. That is,

$$A\mathbf{x_1} = \mathbf{b}, \qquad A\mathbf{x_2} = \mathbf{b}.$$

Let $\mathbf{x_0}$ denote the difference between $\mathbf{x_1}$ and $\mathbf{x_2}$, i.e.

$$\mathbf{x_0} = \mathbf{x_1} - \mathbf{x_2} \neq \mathbf{0}.$$

Consider $A\mathbf{x_0}$: we obtain

$$A\mathbf{x_0} = A(\mathbf{x_1} - \mathbf{x_2}) = A\mathbf{x_1} - A\mathbf{x_2} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

and we conclude that $\mathbf{x_0}$ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
Consider now $A(\mathbf{x_1} + k\mathbf{x_0})$, where k is any real number: we obtain

$$A(\mathbf{x_1} + k\mathbf{x_0}) = A\mathbf{x_1} + Ak\mathbf{x_0} = A\mathbf{x_1} + kA\mathbf{x_0} = \mathbf{b} + k\,\mathbf{0} = \mathbf{b}$$

for any choice of k. We conclude that $\mathbf{x_1} + k\mathbf{x_0}$ gives infinitely many solutions for $A\mathbf{x} = \mathbf{b}$; one solution for every choice of $k \in \mathbb{R}$. Therefore, if any system of the form $A\mathbf{x} = \mathbf{b}$ admits more than one solution, then this system will always admit infnitely many solutions.

2.5 Square systems of linear equations

A square systems of linear equations is a linear system of equations that contains as many equations as unkown variables. For square systems of linear equations the determinant of the coefficient matrix plays a central role for the solutions of the systems. We discuss Cramer's rule, by which certain square systems of linear equations can be solved in terms of determinants.

Theoretical Remarks 2.5.

Consider the square system of linear equations

$$A\,\mathbf{x} = \mathbf{b}, \tag{2.5.1}$$

where A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$.

1. System (2.5.1) can be solved by the use of the Gauss elimination method. See **Theoretical Remark 2.4** for a detailed description of this method.

2. System (2.5.1) admits a unique solution $\mathbf{x} \in \mathbb{R}^n$ for every $\mathbf{b} \in \mathbb{R}^n$ if and only if A is invertible. The unique solution of (2.5.1) is then

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

3. System (2.5.1) admits a unique solution if and only if $\det A \neq 0$. Therefore, if $\det A = 0$ then system (2.5.1) may admit infinitely many solutions or the system may be inconsistent.

4. If system (2.5.1) is consistent, then its unique solution can be calculated by the use of Cramer's rule, which states the following:

Cramer's rule:
If $\det A \neq 0$ then the unique solution $\mathbf{x} = (x_1, x_2, \dots x_n)$ of (2.5.1) is given by the formula

$$x_j = \frac{\det A_j(\mathbf{b})}{\det A}, \quad j = 1, 2, \dots, n,$$

where $A_j(\mathbf{b})$ is the matrix that has been obtained from matrix A by replacing the j^{th} column in A by the vector $\mathbf{b}$. In the case where $\det A = 0$, Cramer's rule states the following:
If $\det A = 0$ and $\det A_j(\mathbf{b}) \neq 0$ for at least one j, then the system (2.5.1) is inconsistent. If $\det A = 0$ and $\det A_j(\mathbf{b}) = 0$ for every $j = 1, 2, \dots, n$, then the system (2.5.1) admits infinitely many solutions.

Problem 2.5.1.

Consider the square system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} k & 1 & 2 \\ 2 & 1 & k \\ k & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -7 \\ 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that the given system has a unique solution. For which values of k is the matrix A invertible?

b) Find all values of k, such that the given system admits infinitely many solutions and give all values of k for which the system is inconsistent.

Solution 2.5.1.

a) We recall that a linear square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det A \neq 0$ and then A is invertible. Calculating the determinant for A, we obtain

$$\det A = \begin{vmatrix} k & 1 & 2 \\ 2 & 1 & k \\ k & 0 & 1 \end{vmatrix} = (k+1)(k-2).$$

Thus the linear system has a unique solution for all $k \in \mathbb{R} \backslash \{-1,\ 2\}$ and this unique solution is

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for all $k \in \mathbb{R} \backslash \{-1,\ 2\}$.

b) To find the values of k for which the linear system $A\mathbf{x} = \mathbf{b}$ might admit infinitely many solutions, we have to investigate the two cases $k = -1$ and $k = 2$, since the determinant of A is zero for those two values of k.

For $k = -1$, the augmented matrix of the system is

$$\begin{pmatrix} -1 & 1 & 2 & 1 \\ 2 & 1 & -1 & -7 \\ -1 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the third row of the previous matrix it is clear that the system is inconsistent in this case, i.e. for $k = -1$.

For $k = 2$, the augmented matrix of the system is

$$\begin{pmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & -7 \\ 2 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 8 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

By the second row of the previous matrix it is clear that the system is also inconsistent in this case ($k = 2$).

We recall that the given system has a unique solution for all $k \in \mathbb{R}\backslash\{-1,\ 2\}$ and that the system is inconsistent for $k = -1$ as well as for $k = 2$. Therefore there exist no real value of k for which the system may admit infinitely many solutions.

Problem 2.5.2.

Consider the following linear system:

$$\begin{pmatrix} 1 & k & 1 \\ k & 1 & 1 \\ -3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k \\ 2 \\ -2 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that the given linear system admits a unique solution and find then this solution by the use of Cramer's rule.

b) Find all values of k, such that the given linear system is inconsistent and all values of k for which it admits infinitely many solutions. Find all solutions.

Solution 2.5.2.

a) We are given the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & k & 1 \\ k & 1 & 1 \\ -3 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} k \\ 2 \\ -2 \end{pmatrix}.$$

The determinant of A is

$$\det A = k^2 - 3k + 2 = (k-1)(k-2)$$

so that A is invertible if and only if $k \in \mathbb{R}\backslash\{1,\ 2\}$. The system has therefore a unique solution for all real k, except for $k = 1$ or $k = 2$. To find this unique solution we use Cramer's rule and calculate

$$x_j = \frac{\det A_j(\mathbf{b})}{\det A}, \quad j = 1, 2, 3.$$

We obtain

$$\det A_1(\mathbf{b}) = \begin{vmatrix} k & k & 1 \\ 2 & 1 & 1 \\ -2 & 0 & -1 \end{vmatrix} = 2 - k$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 1 & k & 1 \\ k & 2 & 1 \\ -3 & -2 & -1 \end{vmatrix} = (k-2)(k-3)$$

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & k & k \\ k & 1 & 2 \\ -3 & 0 & -2 \end{vmatrix} = 2k^2 - 3k - 2 = (2k+1)(k-2).$$

Thus the unique solution of the given system is $\mathbf{x} = (x_1, x_2, x_3)$, where

$$x_1 = -\frac{1}{k-1}, \quad x_2 = \frac{k-3}{k-1}, \quad x_3 = \frac{2k^2 - 3k - 2}{(k-1)(k-2)} = \frac{(2k+1)(k-2)}{(k-1)(k-2)} = \frac{2k+1}{k-1}.$$

Here k is any real number, except $k = 1$ or $k = 2$. For $k = 2$, we note that

$$\det A = 0 \quad \text{and} \quad \det A_j(\mathbf{b}) = 0 \text{ for } j = 1,\ 2,\ 3.$$

Therefore, by Cramer's rule, the system admits infinitely many solutions for $k = 2$. For $k = 1$, we have

$$\det A = 0 \quad \text{and} \quad \det A_j(\mathbf{b}) \neq 0 \text{ for } j = 1,\ 2,\ 3.$$

Therefore, by Cramer's rule, the system is inconsistent for $k = 1$.

b) The augmented matrix for the given system is

$$[A\ \mathbf{b}] = \begin{pmatrix} 1 & k & 1 & k \\ k & 1 & 1 & 2 \\ -3 & 0 & -1 & -2 \end{pmatrix}.$$

We multiply the first row by -1 and add this to the second row, to obtain the row equivalent matrix

$$\begin{pmatrix} 1 & k & 1 & k \\ k-1 & 1-k & 0 & 2-k \\ -3 & 0 & -1 & -2 \end{pmatrix},$$

from which it is clear that the system is inconsisent if and only if $k = 1$. For $k = 2$ the above augmented matrix has the following reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The 3^{rd} column is not a pivot column and we can therefore choose x_3 arbitrary. We let $x_3 = t$. Hence the solutions of the given system for $k = 2$ are

$$\mathbf{x} = -\frac{t}{3}\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.$$

Problem 2.5.3.

Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & k \\ 2k & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Consider the homogeneous linear system

$$A\mathbf{x} = \mathbf{0},$$

where $\mathbf{x} \in \mathbb{R}^4$. Find all values of k, such that this system admits only the trivial (zero) solution, as well as all values of k for which the system admits infinitely many solutions.

b) Consider the non-homogeneous linear system

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Find all values of k, such that this system admits infinitely many solutions $\mathbf{x} \in \mathbb{R}^4$ and give all those solutions. Find also all values of k for which the system admits a unique solution, as well as all k for which the system is inconsistent.

Solution 2.5.3.

a) We recall that the square system $A\mathbf{x} = \mathbf{0}$ admits only the zero solution, $\mathbf{x} = \mathbf{0}$, if and only if A is an invertible matrix. Moreover, A is invertible if and only if $\det A \neq 0$. We therefore calculate the determinant of A:

$$\det A = \begin{vmatrix} 1 & 1 & 0 & k \\ 2k & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} - \begin{vmatrix} 2k & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} - k \begin{vmatrix} 2k & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2k^2 - k + 1$$
$$= -(k+1)(2k-1).$$

From the above we conclude that $A\mathbf{x} = \mathbf{0}$ admits only the zero solution for all $k \in \mathbb{R} \backslash \{-1,\ 1/2\}$ and that the system admits infinitely many solutions for $k = -1$ as well as for $k = 1/2$.

b) Since A is invertible for all $k \in \mathbb{R} \backslash \{-1,\ 1/2\}$, the system has a unique solution for all those values of k and the unique solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Since $\det A = 0$ for $k = -1$ and for $k = 1/2$, we need to investigate the given system $A\mathbf{x} = \mathbf{b}$ for those two values of k. We let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

For $k = -1$ the augmented matrix and its reduced echelon form are as follows:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -1/3 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the 4th column of the augmented matrix is not pivot, we set $x_4 = t$, where t is an arbitrary parameter. Then the solutions are

$$\mathbf{x} = \frac{t}{3} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.$$

For $k = 1/2$ the augmented matrix and its reduced echelon form are as follows:

$$\begin{pmatrix} 1 & 1 & 0 & 1/2 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the 2nd column of the augmented matrix is not pivot, we set $x_2 = t$, where t is an arbitrary parameter. Then the solutions are

$$\mathbf{x} = t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.$$

We conclude that the system has infinitely many solutions for both $k = 1$ and for $k = 1/2$ and it has a unique solution for all other values of k, so that there exist no values of k for which the system is inconsistent.

2.6 Systems of linear equations in $\mathbb{R}^3$

In this section we study systems of linear equations that contain at most three variables. Geometrically these equations are planes in $\mathbb{R}^3$, as discussed in **Chapter 1**. We make use of the method of Gauss elimination, the determinant of a square matrix, and our knowldge of planes that we have gained in **Chapter 1**, in order to solve such systems and to interpret their solutions geometrically in $\mathbb{R}^3$.

Theoretical Remarks 2.6.

The general equation of a plane in $\mathbb{R}^3$ is

$$ax + by + cz = d, \tag{2.6.1}$$

where a, b, c and d are given real numbers. All points (x, y, z) in $\mathbb{R}^3$ which lie on this plane must satisfy equation (2.6.1). Consider now m planes in $\mathbb{R}^3$ given, respectively, by the following system of m equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= d_1 \\ a_{21}x + a_{22}y + a_{23}z &= d_2 \\ &\vdots \\ a_{m1}x + a_{m2}y + a_{m3}z &= d_m. \end{aligned}$$

This system of equations can conveniently be written in the form of a matrix equation

$$A\mathbf{x} = \mathbf{d}, \tag{2.6.2}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

1. Given a finite number of planes in $\mathbb{R}^3$, there exist only four possibilities regarding their common intersection, namely

   a) the planes all intersect in a common point;
   b) the planes all intersect along a common line;
   c) the planes do not all intersect in a common point or along a common line;
   d) the planes all coincide.

   Remark: *If we are given only two planes ($m = 2$), then case a) is not possible.*

2. Corresponding to the above four possibilities for the intersection of m planes, we have the following possibilities for the solutions $\mathbf{x} \in \mathbb{R}^3$ of system (2.6.2):

   a) if the planes all intersect in a common point, then system (2.6.2) has a unique solution;
   b) if the planes all intersect along a common line, then system (2.6.2) admits infinitely many solutions with one free parameter;
   c) if the planes do not all intersect in a common point or along a common line, then the system is inconsistent and has no solutions;
   d) if the planes all coincide, then the system has infinitely many solutions with two free parameters.

   Remark: *If we are given only two planes, then system (2.6.2) with $m = 2$ has either no solutions (no intersection), infinitely many solutions with one free parameter (intersection along a line), or infinitely many solutions with two free parameters (the two planes coincide).*

Problem 2.6.1.

Consider the following three planes in $\mathbb{R}^3$:

$$\begin{aligned} &x - 4y + 7z = 1 \\ &3y - 5z = 0 \\ &-2x + 5y - 9z = k, \end{aligned}$$

where k is an unspecified real parameter. Find all values of k, such that the given three planes intersect along a common line ℓ and give ℓ in parameteric form. Does there exist values of k for which the three planes intersect in a common point? Explain.

Solution 2.6.1.

a) We first write the given equations,

$$\begin{aligned} x - 4y + 7z &= 1 \\ 3y - 5z &= 0 \\ -2x + 5y - 9z &= k \end{aligned}$$

as a matrix equation, namely

$$A\mathbf{x} = \mathbf{b}, \quad \text{where } A = \begin{pmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}.$$

The augmented matrix is

$$[A\ \mathbf{b}] = \begin{pmatrix} 1 & -4 & 7 & 1 \\ 0 & 3 & -5 & 0 \\ -2 & 5 & -9 & k \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 7 & 1 \\ 0 & 3 & -5 & 0 \\ 0 & -3 & 5 & k+2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/3 & 1 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & k+2 \end{pmatrix}.$$

From the above reduced echelon form of $[A\ \mathbf{b}]$, we conclude that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $k = -2$ and for this value of k the system admits infinitely many solutions. Choosing $z = t$ as an arbitrary parameter, these solutions are

$$x = -\frac{1}{3}t + 1, \quad y = \frac{5}{3}t, \quad z = t \quad \text{for all } t \in \mathbb{R}.$$

Thus for all $k \in \mathbb{R}\backslash\{-2\}$ the three planes intersect along a common line ℓ, namely

$$\ell : \begin{cases} x = -\frac{1}{3}t + 1 \\ y = \frac{5}{3}t \\ z = t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Figure 2.1 depicts the intersection of the three planes along the line ℓ for $k = -2$.

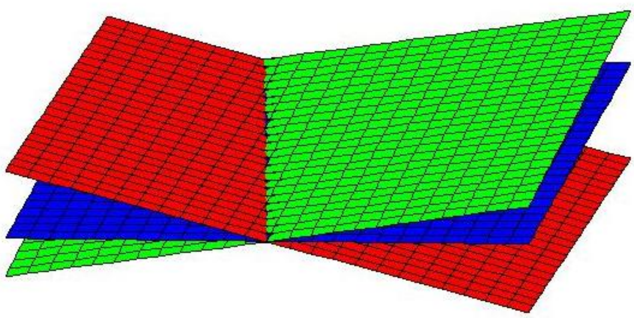

Figure 2.1: Intersection of the planes in Problem 2.6 a) along the line ℓ for $k = -2$.

For all values $k \in \mathbb{R}\backslash\{-2\}$, the system is inconsistent. Figure 2.2 depicts the three planes for the case $k = 6$ and we see that the planes do not intersect along a common line or in a common point.

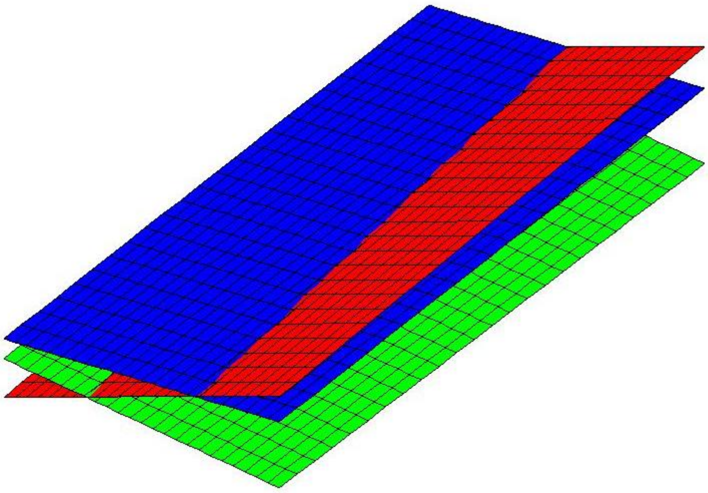

Figure 2.2: No common intersection of the planes in Problem 2.6 a) for $k = 6$.

We conclude that there exists no value of k for which the system admits a unique solution. In other words, there exists no vlaues of k for which the three planes intersect in a common point.

Problem 2.6.2.

Consider the following four planes in $\mathbb{R}^3$:

$$\begin{aligned} x + y &= 2 \\ y + z &= 2 \\ x + z &= 2 \\ ax + by + cz &= 0, \end{aligned}$$

where a, b and c are unspecified real parameters.

a) Find the condition on the parameters a, b and c, such that all four planes intersect in a common point and determine this point under your condition.

b) Find the condition on the parameters a, b and c, such that the given system of four equations is inconsistent. Give the geometrical interpretation of this case in terms of the intersection of the planes.

c) Does there exist any values of the parameters a, b and c for which the four given planes intersect along a common line? Explain.

Solution 2.6.2.

a) We first write the given equations,

$$\begin{aligned} x + y &= 2 \\ y + z &= 2 \\ x + z &= 2 \\ ax + by + cz &= 0, \end{aligned}$$

in the form of a matrix equation, namely

$$A\mathbf{x} = \mathbf{b}, \quad \text{where } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & c \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$[A\ \mathbf{b}] = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{pmatrix}.$$

To write this augmented matrix in its reduced echelon form, it is convenient to concentrate first of all on the first three rows in $[A\ \mathbf{b}]$ and then deal with the parameters in row four. We obtain

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ a & b & c & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{pmatrix}.$$

From the fourth row of the previous matrix we conclude that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the following condition is satisfied:

$$a + b + c = 0$$

and, under this condition, the unique solution of the system is

$$x = 1, \quad y = 1, \quad z = 1.$$

Thus the common point of intersection of the four given planes is $(1, 1, 1)$ for all values of a, b and c, such that $a + b + c = 0$.

b) The given system $A\mathbf{x} = \mathbf{b}$, as described above, is inconsistent for all values of a, b and c, such that

$$a + b + c \neq 0.$$

This means that, for all those values of a, b and c, the four planes will not intersect in a common point or along a common line.

c) As already concluded above, the system admits a unique solution for all those values of a, b and c which satisfy the condition $a + b + c = 0$ and the system is inconsistent for all other values of a, b and c. Thus there exist no values of a, b and c which allow infinitely many solutions for the system, so that there exist no values for which the four planes can intersect along a common line.

Problem 2.6.3.

Consider the following four planes in $\mathbb{R}^3$:

$$\begin{aligned} \Pi_1 : &\quad 2x + 4y + 2z = 12s \\ \Pi_2 : &\quad 2x + 12y + 7z = 12s + 7 \\ \Pi_3 : &\quad x + 10y + 6z = 7s + 8 \\ \Pi_4 : &\quad x + 2y + 3z = -1, \end{aligned}$$

where s is an unspecified real parameter.

a) Find all the values of s, such that the first three planes, Π_1, Π_2 and Π_3, intersect along a common line and present this line of intersection in parametric form.

b) Find the common point of intersection of all four planes, Π_1, Π_2, Π_3 and Π_4, if such a point exists.

Solution 2.6.3.

a) The planes Π_1, Π_2 and Π_3 can be written in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 2 & 12 & 7 \\ 1 & 10 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 12s \\ 12s+7 \\ 7s+8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We now write the augmented matrix $[A\ \mathbf{b}]$ in its reduced echelon form:

$$[A\ \mathbf{b}] = \begin{pmatrix} 2 & 4 & 2 & 12s \\ 2 & 12 & 7 & 12s+7 \\ 1 & 10 & 6 & 7s+8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/4 & (24s-7)/4 \\ 0 & 1 & 5/8 & 7/8 \\ 0 & 0 & 0 & s+1 \end{pmatrix}.$$

We conclude that this system is consistent if and only if $s = -1$. Then the system reduces to

$$y + \frac{5}{8}z = \frac{7}{8}, \quad x - \frac{1}{4}z = -\frac{31}{4}.$$

Clearly z is an arbitrary parameter, so we let $z = t$. The solution is then

$$x = \frac{1}{4}t - \frac{31}{4}, \quad y = -\frac{5}{8}t + \frac{7}{8}, \quad z = t \quad \text{for all } t \in \mathbb{R},$$

so that the parametric equation for the line ℓ, that describes the intersection of the planes Π_1, Π_2 and Π_3 for $s = -1$, is

$$\ell : \begin{cases} x = \frac{1}{4}t - \frac{31}{4} \\ y = -\frac{5}{8}t + \frac{7}{8} \\ z = t \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

Note that the planes Π_1, Π_2 and Π_3 only intersect along this common line ℓ if $s = -1$. For all other values of s, namely $s \in \mathbb{R}\backslash\{-1\}$, the planes do not intersect along a common line.

b) To find the intersection of Π_4 with the line ℓ obtained in part a) above, we need to find the value of t such that $\Pi_4 :\ x + 2y + 3z = -1$ is satisfied. That is

$$-\frac{31}{4} + \frac{1}{4}t + 2\left(\frac{7}{8} - \frac{5}{8}t\right) + 3t = -1,$$

which gives $t = \frac{5}{2}$. The point of the intersection of Π_1, Π_2, Π_3 and Π_4 is then $(x_1,\ y_1,\ z_1)$, where

$$x_1 = \frac{1}{4}\left(\frac{5}{2}\right) - \frac{31}{4} = -\frac{57}{8}, \quad y_1 = -\frac{5}{8}\left(\frac{5}{2}\right) + \frac{7}{8} = -\frac{11}{16}, \quad z_1 = \frac{5}{2}.$$

In Figure 2.3 we depict the intersection of the four planes in the point $(-\frac{57}{8}, -\frac{11}{16}, \frac{5}{2})$.

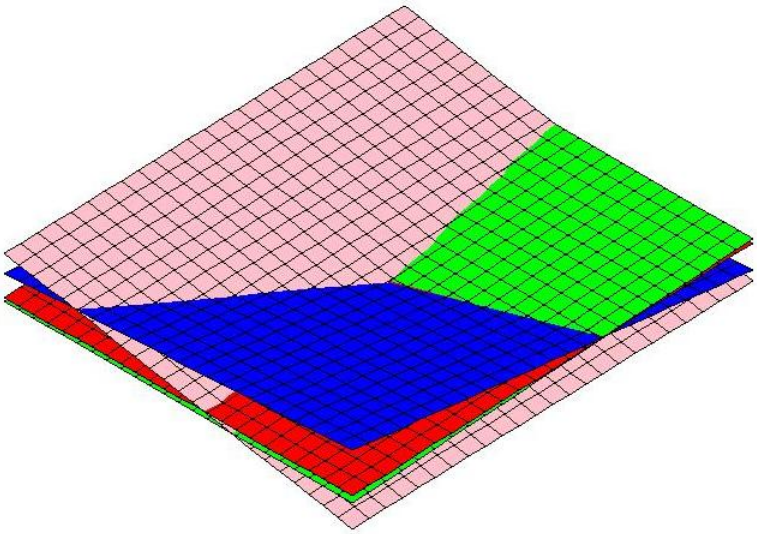

Figure 2.3: The intersection of the four planes in Problem 2.6 c) for $s = -1$.

Problem 2.6.4.

Consider the following six planes which describe a parallelepiped at their intersections (see Figure 2.4):

$$\begin{aligned}
\Pi_1 &: \ x + y - 4z = -10 \\
\Pi_2 &: \ x + y - 4z = -6 \\
\Pi_3 &: \ y - 2z = -2 \\
\Pi_4 &: \ y - 2z = -3 \\
\Pi_5 &: \ x - 3y + 8z = 18 \\
\Pi_6 &: \ x - 3y + 8z = 14.
\end{aligned}$$

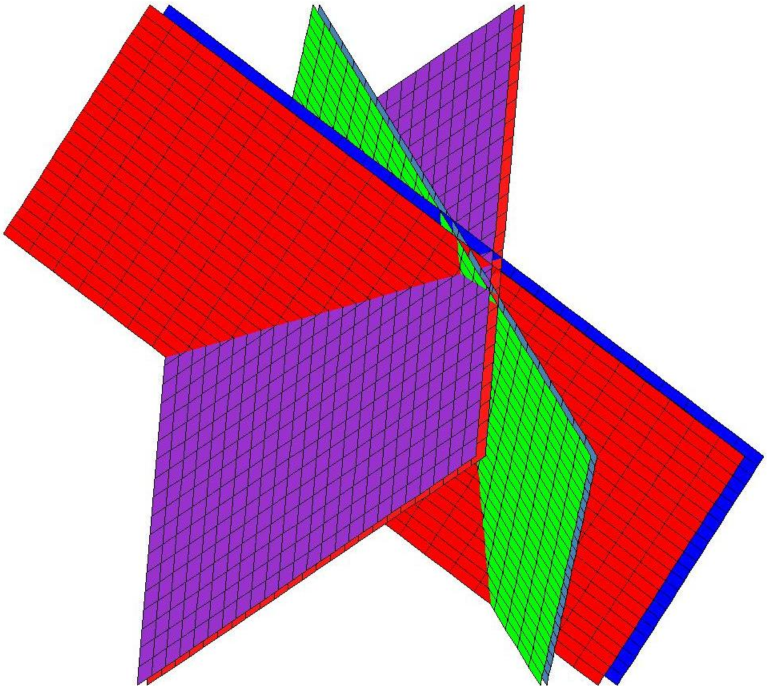

Figure 2.4: Six planes that describe a parallelepiped at their intersections.

Find the vertices, the volume and the midpoint of the parallelepiped.

Solution 2.6.4.

The equations of the planes Π_j and their corresponding normal vectors $\mathbf{n}_j$ $(j = 1, 2, \ldots, 6)$ that describe the six faces of the parallelepiped in Figure 2.5, are as follows:

$$\begin{aligned}
&\Pi_1 : \ x + y - 4z = -10, \quad \mathbf{n}_1 = (1, 1, -4)\\
&\Pi_2 : \ x + y - 4z = -6 \quad \mathbf{n}_2 = (1, 1, -4)\\
&\Pi_3 : \ y - 2z = -2, \quad \mathbf{n}_3 = (0, 1, -2)\\
&\Pi_4 : \ y - 2z = -3 \quad \mathbf{n}_4 = (0, 1, -2)\\
&\Pi_5 : \ x - 3y + 8z = 18, \quad \mathbf{n}_5 = (1, -3, 8)\\
&\Pi_6 : \ x - 3y + 8z = 14, \quad \mathbf{n}_6 = (1, -3, 8).
\end{aligned}$$

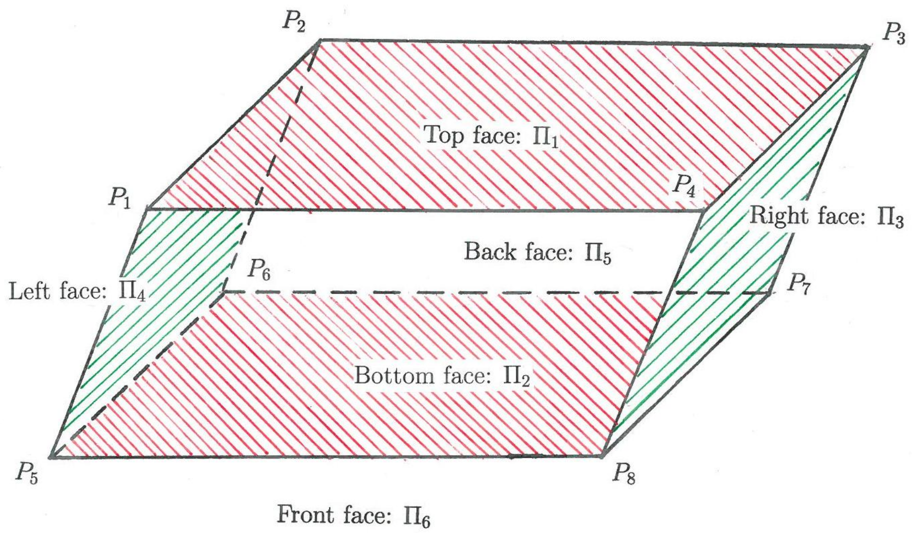

Figure 2.5: The parallelepiped enclosed by the six planes Π_1, Π_2, Π_3, Π_4, Π_5 and Π_6.

The coordinates of the vertice $P_1 : (x_1, y_1, z_1)$ is given by the intersection of planes Π_1, Π_4 and Π_6 (see Figure 2.5). We write

$$P_1 : \ \Pi_1 \cap \Pi_4 \cap \Pi_6.$$

This is obtained by the unique solution of the linear system

$$A_1 \mathbf{x_1} = \mathbf{b_1},$$

where

$$A_1 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_1} = \begin{pmatrix} -10 \\ -3 \\ 14 \end{pmatrix}.$$

We note that $\det A_1 = 4$, which ensures the existence of a unique solution for the above linear system, namely the solution $\mathbf{x_1} = A_1^{-1}\mathbf{b_1}$. We obtain

$$\mathbf{x_1} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}.$$

The coordinates of the vertice $P_2 : (x_2, y_2, z_2)$ is given by the following intersecting planes (see Figure 2.5):

$$P_2 : \ \Pi_1 \cap \Pi_4 \cap \Pi_5.$$

This is obtained by the unique solution of the linear system

$$A_2\mathbf{x_2} = \mathbf{b_2},$$

where

$$A_2 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_2} = \begin{pmatrix} -10 \\ -3 \\ 18 \end{pmatrix}.$$

Now $\det A_2 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_2} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}.$$

The coordinates of the vertice $P_3 : (x_3, y_3, z_3)$ is given by the following intersecting planes (see Figure 2.5):

$$P_3 : \ \Pi_1 \cap \Pi_3 \cap \Pi_5.$$

This is obtained by the unique solution of the linear system

$$A_3\mathbf{x_3} = \mathbf{b_3},$$

where

$$A_3 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_3} = \begin{pmatrix} -10 \\ -2 \\ 18 \end{pmatrix}.$$

Now $\det A_3 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_3} = \begin{pmatrix} 2 \\ 8 \\ 5 \end{pmatrix}.$$

The coordinates of the vertice $P_4 : (x_4, y_4, z_4)$ is given by the following intersecting planes (see Figure 2.5):

$$P_4 : \ \Pi_1 \cap \Pi_3 \cap \Pi_6.$$

This is obtained by the unique solution of the linear system

$$A_4\mathbf{x_4} = \mathbf{b_4},$$

where

$$A_4 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_4} = \begin{pmatrix} -10 \\ -2 \\ 14 \end{pmatrix}.$$

Now $\det A_4 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_4} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}.$$

The coordinates of the vertice $P_5 : (x_5, y_5, z_5)$ is given by the following intersecting planes (see Figure 2.5):

$$P_5 : \ \Pi_2 \cap \Pi_4 \cap \Pi_6.$$

This is obtained by the unique solution of the linear system

$$A_5\mathbf{x_5} = \mathbf{b_5},$$

where

$$A_5 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_5} = \begin{pmatrix} -6 \\ -3 \\ 14 \end{pmatrix}.$$

Now $\det A_5 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_5} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The coordinates of the vertice $P_6 : (x_6, y_6, z_6)$ is given by the following intersecting planes (see Figure 2.5):

$$P_6 : \ \Pi_2 \cap \Pi_4 \cap \Pi_5.$$

This is obtained by the unique solution of the linear system

$$A_6\mathbf{x_6} = \mathbf{b_6},$$

where

$$A_6 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_6} = \begin{pmatrix} -6 \\ -3 \\ 18 \end{pmatrix}.$$

Now $\det A_6 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_6} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

The coordinates of the vertice $P_7 : (x_7, y_7, z_7)$ is given by the following intersecting planes (see Figure 2.5):

$$P_7 : \ \Pi_2 \cap \Pi_3 \cap \Pi_5.$$

This is obtained by the unique solution of the linear system

$$A_7\mathbf{x_7} = \mathbf{b_7},$$

where

$$A_7 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_7} = \begin{pmatrix} -6 \\ -2 \\ 18 \end{pmatrix}.$$

Now $\det A_7 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_7} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix}.$$

The coordinates of the vertice $P_8 : (x_8, y_8, z_8)$ is given by the following intersecting planes (see Figure 2.5):

$$P_8 : \ \Pi_2 \cap \Pi_3 \cap \Pi_6.$$

This is obtained by the unique solution of the linear system

$$A_8\mathbf{x_8} = \mathbf{b_8},$$

where

$$A_8 = \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & -3 & 8 \end{pmatrix}, \qquad \mathbf{b_8} = \begin{pmatrix} -6 \\ -2 \\ 14 \end{pmatrix}.$$

Now $\det A_8 = 4$, which again ensures the existence of a unique solution for this linear system. We obtain

$$\mathbf{x_8} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

We sum up: the coordinates of the vertices of the parallelepiped are as follows (see Figure 2.5):

$$P_1 : \ (-1, 3, 3) \quad P_2 : \ (1, 5, 4), \quad P_3 : \ (2, 8, 5), \quad P_4 : \ (0, 6, 4)$$

$$P_5 : \ (1, 1, 2), \quad P_6 : \ (3, 3, 3), \quad P_7 : \ (4, 6, 4), \quad P_8 : \ (2, 4, 3).$$

The volume V of the above parallelepiped is given by the following scalar triple product

$$V = |\left(\overrightarrow{P_5P_8} \times \overrightarrow{P_5P_6}\right) \cdot \overrightarrow{P_5P_1}\ |,$$

where $|\ \ |$ denotes the absolute value and

$$\overrightarrow{P_5P_8} = (1,3,1), \quad \overrightarrow{P_5P_6} = (2,2,1), \quad \overrightarrow{P_5P_1} = (-2,2,1).$$

We obtain

$$V = |\begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ -2 & 2 & 1 \end{vmatrix}| = |-4| = 4 \text{ cubic units}$$

To find the midpoint $Q:\ (x,y,z)$ of the parallelepiped we can consider, for example, the vertices P_5 and P_3, where

$$\overrightarrow{P_5Q} = \frac{1}{2}\overrightarrow{P_5P_3}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}.$$

The coordinates of the the midpoint Q is therefore $Q:\ (\frac{3}{2}, \frac{9}{2}, \frac{7}{2})$.

To find the hight h of the parallelepiped with base plane Π_2, we project the vector $\overrightarrow{P_5P_1}$ orthogonally onto the normal vector $\mathbf{n_2} = (1,1,-4)$ of Π_2, i.e.

$$h = \|\text{proj}_{\mathbf{n_2}}\, \overrightarrow{P_5P_1}\| = \|\left(\frac{\overrightarrow{P_5P_1} \cdot \mathbf{n_2}}{\mathbf{n_2} \cdot \mathbf{n_2}}\right)\mathbf{n_2} = \|(-\frac{2}{9}, \frac{2}{9}, \frac{8}{9})\| = \frac{2\sqrt{2}}{3}.$$

Alternately, we can calculate the distance s between the planes Π_1 and Π_2, which is given by the relation

$$s = \frac{|d_1 - d_2|}{\|n\|},$$

for $\Pi_1:\ ax + by + cz = d_1$ and $\Pi_2:\ ax + by + cz = d_2$ (see **Theoretical Remark 1.5**). In our case we have

$$a = 1, \quad b = 1, \quad c = -4, \quad d_1 = -10, \quad d_2 = -6, \quad \mathbf{n} = (1,1,-4).$$

This leads to $s = \dfrac{|-10-(-6)|}{\sqrt{1+1+16}} = \dfrac{4}{\sqrt{18}} = \dfrac{2\sqrt{2}}{3}$.

2.7 Intersection of lines in $\mathbb{R}^3$

In this section we discuss intersections of lines in $\mathbb{R}^3$ and show how to calculate those intersections.

Theoretical Remarks 2.7.

Given two lines in $\mathbb{R}^3$, say ℓ_1 and ℓ_2, we have the following possibilities regarding their intersection:

a) ℓ_1 and ℓ_2 may intersect in a unique common point.

b) ℓ_1 and ℓ_2 may intersect at every point on ℓ_1 and ℓ_2, so that the two lines coincide.

c) ℓ_1 and ℓ_2 may not intersect at any point.

Remark: *Given any number of lines in $\mathbb{R}^3$, the possibilities of their common intersections are the same as those listed above for two lines.*

Problem 2.7.1.

Consider the following two lines in $\mathbb{R}^3$:

$$\ell_1 : \begin{cases} x = 2t+3 \\ y = -4t+1 \\ z = 2t+2 \end{cases} \qquad \ell_2 : \begin{cases} x = -s \\ y = bs+3 \\ z = -s-1 \end{cases}$$

for all $t \in \mathbb{R}$ and all $s \in \mathbb{R}$, where b is an unspecified real parameter.

a) Find all values of b, such that the lines ℓ_1 and ℓ_2 intersect.

b) Do the lines intersect for $b = 1$? If so, find the intersection(s) in this case.

Solution 2.7.1.

a) At any point where ℓ_1 and ℓ_2 intersect, there must exist parametric values for t and s for the coordinates of the intersection points. To establish those points, we consider

$$\begin{aligned} x &= 2t+3 = -s \\ y &= -4t+1 = bs+3 \\ z &= 2t+2 = -s-1. \end{aligned}$$

We have

$$\begin{aligned} 2t+3 &= -s \\ -4t+1 &= bs+3 \\ 2t+2 &= -s-1 \end{aligned}$$

and in matix form we have

$$\begin{pmatrix} 2 & 1 \\ -4 & -b \\ 2 & 1 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}.$$

The corresponding augmented matrix is

$$\begin{pmatrix} 2 & 1 & -3 \\ -4 & -b & 2 \\ 2 & 1 & -3 \end{pmatrix}.$$

We now apply two elementary row operations to the above augmented matrix, namely

1: multiply the first row by 2 and add the resulting row to the second row;
2: multiply the first row by -1 and add the resulting row to the third row.
This leads to the following echelon form:

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & 2-b & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above echelon form we conclude that the system has a solution for t and s if and only if

$$b \neq 2.$$

Therefore the two lines ℓ_1 and ℓ_2 intersect if and only if

$$b \in \mathbb{R}\backslash\{2\}.$$

For those values of b, we have

$$t = \frac{2-3b}{2(b-2)}, \qquad s = \frac{4}{b-2}.$$

Inserting the above values for t and s into ℓ_1 or ℓ_2, we obtain the x-, y- and z-coordinates of the point of intersection for any $b \in \mathbb{R}\backslash\{2\}$, namely

$$x = \frac{-4}{b-2}, \qquad y = \frac{7b-6}{b-2}, \qquad z = -\frac{b+2}{b-2}.$$

b) For $b = 1$ the lines ℓ_1 and ℓ_2 intersect and the parameteric values of t and s are (see above)

$$t = \frac{1}{2}, \qquad s = -4.$$

Inserting those values for t and s into ℓ_1 or ℓ_2 we obtain the coordinates of the point of intersection, namely

$$(4, -1, 3).$$

Problem 2.7.2.

Consider the following three lines in $\mathbb{R}^3$:

$$\ell_1 : \begin{cases} x = -t+3 \\ y = 2t+1 \\ z = -t+2 \end{cases} \qquad \ell_2 : \begin{cases} x = 3s+3 \\ y = -6s+1 \\ z = 3s+2 \end{cases} \qquad \ell_3 : \begin{cases} x = -4p+8 \\ y = p+2 \\ z = 2p-1 \end{cases}$$

for all $t \in \mathbb{R}$, all $s \in \mathbb{R}$ and all $p \in \mathbb{R}$.

a) Find the intersection(s) of the lines ℓ_1 and ℓ_2, if those lines do intersect.

b) Find the intersection(s) of the lines ℓ_1, ℓ_2 and ℓ_3, if those lines do intersect.

Solution 2.7.2.

a) To find the intersection(s) of ℓ_1 and ℓ_2 we consider

$$\begin{aligned} x &= -t+3 = 3s+3 \\ y &= 2t+1 = -6s+1 \\ z &= -t+2 = 3s+2, \end{aligned}$$

so that

$$\begin{aligned} -t-3s &= 0 \\ 2t+6s &= 0 \\ -t-3s &= 0. \end{aligned}$$

In matrix form we have

$$\begin{pmatrix} -1 & -3 \\ 2 & 6 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the corresponding augmented matrix we have

$$\begin{pmatrix} -1 & -3 & 0 \\ 2 & 6 & 0 \\ -1 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which means that

$$t = -3s \quad \text{for all } s \in \mathbb{R}.$$

Thus for every value of $s \in \mathbb{R}$ for ℓ_2 there is a value of t for ℓ_1, namely $t = -3s$, that gives the same coordinates and hence a point of intersection between ℓ_1 and ℓ_2. The two lines, ℓ_1 and ℓ_2, therefore intersect at every point on ℓ_1 (or ℓ_2), so that the two lines in fact coincide.

b) Since ℓ_1 and ℓ_2 coincide (as established above), we can now search for the intersection(s) between ℓ_1 and ℓ_3. We consider

$$\begin{aligned} x &= -t+3 = -4p+8 \\ y &= 2t+1 = p+2 \\ z &= -t+2 = 2p-1, \end{aligned}$$

so that the matrix equation takes the form

$$\begin{pmatrix} -1 & 4 \\ 2 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} t \\ p \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}.$$

For the corresponding augmented matrix we have

$$\begin{pmatrix} -1 & 4 & 5 \\ 2 & -1 & 1 \\ -1 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -5 \\ 0 & 7 & 11 \\ 0 & -6 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 11/7 \\ 0 & 0 & 1 \end{pmatrix},$$

which means that the system is inconsistent. Hence there exist no values for t and p which would give the same point, so that there is no intersection between ℓ_1 and ℓ_3. The three lines, ℓ_1, ℓ_2 and ℓ_3, do therefore not intersect in a common point or points.

2.8 Exercises

1. Consider the matrix equation

   $$AX^{-1} + 3B = A^2,$$

   where

   $$A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

   and X is an invertible 2×2 matrix. Find X, such that the given matrix equation is satisfied.

   [**Answer:** $X = \begin{pmatrix} -2/17 & -5/17 \\ -1/17 & 6/17 \end{pmatrix}$.]

2. Consider the matrix equation

   $$2X + AX = 3B,$$

   where

   $$A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$$

   and X is an unspecified matrix.

   a) For the given matrix equation, assume that

   $$B = \begin{pmatrix} 3 & 1 \\ -3 & 2 \end{pmatrix}$$

   and find the matrix X.

   [**Answer:** $X = \begin{pmatrix} 9 & -6 \\ -6 & 7 \end{pmatrix}$.]

   b) For the given matrix equation, assume that

   $$B = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

   and find the matrix X.

   [**Answer:** $X = \begin{pmatrix} 3 & -3 & -9 \\ -2 & 2 & 7 \end{pmatrix}$.]

3. Consider the matrix equation

$$C^{-1}(XB - A)B^{-1} = X,$$

where B, C and $I_n - C$ are all $n \times n$ invertible matrices. Find the matrix X that satisfies the above equation in terms of the other given matrices and in terms of the identity matrix I_n.

[**Answer:** $X = (I_n - C)^{-1}AB^{-1}$.]

4. Consider the matrix equation

$$AX^{-1} + (X + B)^{-1} = X^{-1},$$

where $A,\ B,\ X,\ X + B,\ A^{-1} - I_n$ and $A - I_n$ are all $n \times n$ invertible matrices.

a) Solve the given matrix equation for X.

[**Answer:** $X = B(A^{-1} - I_n)$.]

b) Solve the given matrix equation for X, where A and B take the following explicit forms:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

[**Answer:** $X = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.]

5. Consider the following matrix:

$$A = \begin{pmatrix} a & 2 & 3 \\ 1 & 0 & -1 \\ -1 & 3 & a+6 \end{pmatrix},$$

where a is an unspecified real parameter.

a) Find all values of a, such that the matrix A is invertible.

[**Answer:** $a \in \mathbb{R}\backslash\{1\}$.]

b) Calculate the inverse of A with $a = 0$.

[**Answer:** $A^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 5 & -3 & -3 \\ -3 & 2 & 2 \end{pmatrix}$.]

6. Find all solutions of the system $A\mathbf{x} = \mathbf{b}$ for the given matrix A and vector $\mathbf{b}$ as given below. Give also the geometrical interpretation of the solutions where possible.

a) $A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -2 & 0 \\ -2 & 8 & 5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

[**Answer:** The unique solution is given by a point in $\mathbb{R}^3$, the coordinates of which are $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$.]

b) $A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -2 & 0 \\ -2 & 8 & 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$

[**Answer:** The solutions are given by a line in $\mathbb{R}^3$ passing through the point $(3/2,\ 0,\ 1/2)$ and parallel to the vector $(1,\ 1,\ -1)$, i.e. the infinitely many solutions are

$$\mathbf{x} = t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 0 \\ 1/2 \end{pmatrix} \text{ for all } t \in \mathbb{R}.]$$

c) $A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

[**Answer:** The solutions are given by a plane in $\mathbb{R}^3$ with equation $x_1 - x_2 + 2x_3 = 1$, i.e. the infinitely many solutions are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for all } t \in \mathbb{R} \text{ and all } s \in \mathbb{R}.]$$

d) $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ -3 & 3 & -6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

[**Answer:** The system is inconsistent. That is, the system has no solution.]

7. Find the intersection of the following two planes in $\mathbb{R}^3$:

$$\Pi_1: \quad x - y + 3z = 1$$
$$\Pi_2: \quad x + y + 2z = 10.$$

Use Maple to sketch the planes in $\mathbb{R}^3$ (see Appendix A for information about Maple).

[**Answer:** The two planes intersect along the following line:

$$\ell : \begin{cases} x = -5t + 28 \\ y = t \\ z = 2t - 9 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$]

8. Find the intersection of the following three planes in $\mathbb{R}^3$:

$$\begin{aligned}\Pi_1 &: \quad x + 3y - 5z = 0\\ \Pi_2 &: \quad x + 4y - 8z = 0\\ \Pi_3 &: -2x - 7y + 13z = 0.\end{aligned}$$

Use Maple to sketch the planes in $\mathbb{R}^3$ (See Appendix A for information about Maple).

[**Answer:** The three planes intersect along the following line:

$$\ell : \begin{cases} x = -4t \\ y = 3t \\ z = t \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$]

9. Consider the following three planes in $\mathbb{R}^3$:

$$\begin{aligned}\Pi_1 &: \quad x_1 - 4x_2 + 7x_3 = 1\\ \Pi_2 &: \quad 3x_2 - 5x_3 = 0\\ \Pi_3 &: \quad -2x_1 + 5x_2 - 9x_3 = k,\end{aligned}$$

where k is an unspecified real parameter.

a) Find all values of k such that the given three planes intersect along a common line ℓ and give this line of intersection in parametric form.

[**Answer:** The three planes intersect along a common line ℓ if and only if $k = -2$, where ℓ is given by

$$\ell : \begin{cases} x_1 = -t/5 + 1 \\ x_2 = t \\ x_3 = 3t/5 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$]

b) For which value(s) of k do the three planes intersect in a unique point.

[**Answer:** There exists no value of k for which the three planes intersect in a unique point.]

10. Find all solutions of the following system:

$$\begin{aligned} &x_3 + 2x_5 = 1 \\ &x_1 + 6x_2 + 2x_3 + 4x_5 = -1 \\ &x_4 + 5x_5 = 2. \end{aligned}$$

[**Answer:** $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ -2 \\ -5 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$
for all $t \in \mathbb{R}$ and all $s \in \mathbb{R}$.]

11. Consider the following system:

$$\begin{aligned} &x_1 + x_3 + 2x_4 = 1 \\ &2x_1 + kx_2 + x_3 + x_4 = 2 \\ &3x_2 + x_3 + 2x_4 = 3 \\ &x_1 + x_2 + x_4 = 4, \end{aligned}$$

where k is an unspecified real parameter.

a) Find all values of k, such that the given system has a unique solution.

[**Answer:** $k \in \mathbb{R} \backslash \{-7\}$.]

b) Find all values of k, such that the given system has infinitely many solutions.

[**Answer:** There exist no values of k for which the system admits infinitely many solutions.]

c) Find all values of k, such that the given system is inconsistent.

[**Answer:** $k = -7$.]

d) Find all values of k for which the coefficient matrix of the given system is singular.

[**Answer:** $k = -7$.]

12. Consider the following system:

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 3x_1 + kx_3 &= b \\ x_1 + kx_2 + x_3 &= c, \end{aligned}$$

where $a,\ b,\ c$ and k are unspecified real parameters.

a) Find all values of k, such that the given system has a unique solution for all real values of $a,\ b$ and c.

[**Answer:** $k \in \mathbb{R}\backslash\{1,\ 3\}$.]

b) Find all values of k and the corresponding conditions on $a,\ b$ and c, such that the given system is consistent.

[**Answer:** From part a) above, we know that the system has a unique solution (and is consistent) for all $k \in \mathbb{R}\backslash\{1,\ 3\}$ and all real values of $a,\ b$ and c. For $k = 1$ the system has infinitely many solutions (and is consistent) if and only if $c = a$ for all $c \in \mathbb{R}$. For $k = 3$ the system has infinitely many solutions (and is consistent) if and only if $c = 3a - 2b/3$ for all $a \in \mathbb{R}$ and all $b \in \mathbb{R}$.]

13. Consider the following matrix equation:

$$X\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \alpha & -\alpha \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},$$

where X is an unspecified 2×2 matrix. Determine all real values of α, such that the given matrix equation has a unique solution for X.

[**Answer:** $\alpha \in \mathbb{R}\backslash\{-2\}$.]

14. a) Consider the function

$$f(x) = ax^3 + bx^2 + cx + d,$$

where $a,\ b,\ c$ and d are unspecified real parameters. Find the values of these parameters such that the graph $y = f(x)$ is passing through the following points in the xy-plane: $\{(1,1),\ (-1,1),\ (2,2),\ (-2,12)\}$. Use Maple to sketch your obtained function $f(x)$ in the xy-plane (see Appendix A for information about Maple).

[**Answer:** $a = -5/6,\ b = 2,\ c = 5/6,\ d = -1$.]

b) Consider the function

$$f(x) = a\cos(2x) + b(\pi - x)\cos(2x) + cx\sin(\pi - x),$$

where a, b and c are unspecified real parameters. Find the values of these parameters such that the graph $y = f(x)$ is passing through the following points in the xy-plane: $\{(-\pi/2, -3\pi),\ (\pi/2, 0),\ (3\pi/2, 5\pi)\}$. Use Maple to sketch your obtained function $f(x)$ in the xy-plane (see Appendix A for information about Maple).

[**Answer:** $a = -2\pi,\ b = 3,\ c = -1.$]

15. Consider the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ h \\ k \end{pmatrix},$$

where h and k are unspecified real parameters.

a) Find the relation between the parameters h and k, such that the given system $A\mathbf{x} = \mathbf{b}$ is consistent.

[**Answer:** $k = 2h$ for all $h \in \mathbb{R}$.]

b) Find all solutions for the given system $A\mathbf{x} = \mathbf{b}$.

[**Answer:** The system has the unique solution $\mathbf{x} = \begin{pmatrix} h-3 \\ 6-h \end{pmatrix}$ for all $h \in \mathbb{R}$, where $k = 2h$.]

16. Consider the following line ℓ in $\mathbb{R}^3$:

$$\ell : \begin{cases} x = 2t + 1 \\ y = -2t + 1 \\ z = 6t - 6 \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

Find all real values of the parameters a, b and c, such that the line ℓ is lying on the plane

$$ax + by + cz = 1.$$

[**Answer:** $a = \frac{1}{3} + \frac{b}{3}$, $c = -\frac{1}{9} + \frac{2b}{9}$ for all $b \in \mathbb{R}$.]

17. Consider the following two lines in $\mathbb{R}^3$:

$$\ell_1 : \begin{cases} x = 2t + 3 \\ y = -4t + 1 \\ z = 2t + 2 \end{cases} \qquad \ell_2 : \begin{cases} x = -s \\ y = bs + 3 \\ z = -s - 1 \end{cases}$$

for all $t \in \mathbb{R}$ and all $s \in \mathbb{R}$, where b is an unspecified real parameter.

a) Find all values of b, such that the lines ℓ_1 and ℓ_2 intersect.

[**Answer:** $b \in \mathbb{R}\backslash\{2\}$.]

b) Do the lines intersect for $b = 1$? If so, find the point of intersection for this case.

[**Answer:** Yes, the point of intersection has the coordinates $(4, -1, 3)$.]

18. Consider the following six planes that describe a parallelepiped at their intersections:

$$\begin{aligned}
\Pi_1 &: \ x + 2y - z = 1 \\
\Pi_2 &: \ 2x + 4y - 2z = 0 \\
\Pi_3 &: \ -3x - y + 2z = 1 \\
\Pi_4 &: \ -9x - 3y + 6z = 1 \\
\Pi_5 &: \ y + z = -1 \\
\Pi_6 &: \ -2y - 2z = 3.
\end{aligned}$$

Find the vertices, the volume and the midpoint of this parallelepiped, as well as the hight of the parallelepiped with base face described by Π_2.

[**Answer:** The coordinates of the vertices of the parallelepiped are as follows:

$$\begin{aligned}
P_1 &: \ (-\frac{17}{12}, \ \frac{11}{36}, \ -\frac{65}{36}) \\
P_2 &: \ (-\frac{7}{6}, \ \frac{7}{18}, \ -\frac{25}{18}) \\
P_3 &: \ (-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{3}{2}) \\
P_4 &: \ (-\frac{7}{4}, \ \frac{5}{12}, \ -\frac{23}{12}) \\
P_5 &: \ (-\frac{11}{12}, \ -\frac{7}{36}, \ -\frac{47}{36}) \\
P_6 &: \ (-\frac{2}{3}, \ -\frac{1}{9}, \ -\frac{8}{9}) \\
P_7 &: \ (-1, \ 0, \ -1) \\
P_8 &: \ (-\frac{5}{4}, \ -\frac{1}{12}, \ -\frac{17}{12})
\end{aligned}$$

The volume of the parallelepiped is 1/18 cubic units. The coordinates of the the midpoint Q of the parallelepiped is

$$Q: \ (-\frac{29}{24}, \frac{11}{72}, -\frac{101}{72}).$$

The hight of the parallelepiped with base described by Π_2 is $1/\sqrt{6}$ units.]

19. Consider the following two planes:

$$\Pi_1: \ x + 2y - 4z = 2$$
$$\Pi_2: \ x - z = 5.$$

Find the equation of the planes Π_1^*, such that Π_1^* is the reflection of the plane Π_1 about the plane Π_2.

[**Answer:** $\Pi_1^*: \ 4x - 2y - z = 23.$]

20. Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$\begin{pmatrix} 1 & 3 & k \\ k & 1 & 4 \\ 1 & k & k \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that the given system admits a unique solution.

[**Answer:** The system admits a unique solution for all $k \in \mathbb{R}\setminus\{-2, \ 2, \ 3\}$.]

b) Find all values of k, such that the given system admits infinitely many solutions, as well as all values of k for which the system is inconsistent.

[**Answer:** For $k = -2$ the system has infinitely many solutions, namely $\mathbf{x} = t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ for all $t \in \mathbb{R}$. For $k = 2$ as well as for $k = 3$ the system is inconsistent.]

c) Find all values of k, such that the coefficient matrix A is singular.

[**Answer:** The matrix A is singular if and only if $\det A = 0$, that is, A is singular for $k \in \{-2, \ 2, \ 3\}$.]

21. The following three lines, ℓ_1, ℓ_2 and ℓ_3, describe a triangle in $\mathbb{R}^3$ at their intersections:

$$\ell_1 : \begin{cases} x = 4\alpha - 1 \\ y = -2\alpha + 3 \\ z = 8\alpha - 3 \quad \text{for all } \alpha \in \mathbb{R} \end{cases} \qquad \ell_2 : \begin{cases} x = -3\beta + 7 \\ y = -\beta + 4 \\ z = -\beta + 3 \quad \text{for all } \beta \in \mathbb{R} \end{cases}$$

$$\ell_3 : \begin{cases} x = -\delta + 6 \\ y = -2\delta + 7 \\ z = 3\delta - 4 \quad \text{for all } \delta \in \mathbb{R}. \end{cases}$$

Find the area of this triangle.

[**Answer:** The area of the triangle is $5\sqrt{6}$ square units.]

22. Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 5 & 1 & k \\ 2 & 1 & k & 1 \\ 1 & 4 & 1 & 1 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

and k is an unspecified real parameter.

a) Find all values of k, such that the system admits only the trivial solution and all values of k for which A is invertible.

[**Answer:** For all $k \in \mathbb{R} \setminus \{\frac{4}{3}, \frac{7}{5}\}$ the system admits only the trivial solution $\mathbf{x} = (0, 0, 0, 0)$. The matrix A is also invertible for those values of k.]

b) Find all values of k, such that the system admits infinitely many solutions.

[**Answer:** For all $k = \frac{4}{3}$ or $k = \frac{7}{5}$ the system admits infinitely many solutions.]

Chapter 3

Spanning sets and linearly independent sets

The aim of this chapter:

In this chapter we introduce the following definitions and concepts for a finite set of vectors in $\mathbb{R}^n$: *linear combinations of vectors*, *spanning sets* and *linearly independent sets of vectors*. We apply these concepts to describe, for example, a plane or a line in $\mathbb{R}^3$ and to gain a better understanding of linear systems.

3.1 Linear combinations of vectors

In this section we introduce the concept of a linear combination for a finite set of vectors in $\mathbb{R}^n$.

Theoretical Remarks 3.1.

Consider the set S of p vectors

$$S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}\},$$

where $\mathbf{u_j} \in \mathbb{R}^n$ for $j = 1, 2, \ldots, p$.

1. A **linear combination** of the vectors from the set S is another vector in $\mathbb{R}^n$, namely the vector

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_p\mathbf{u_p} \in \mathbb{R}^n$$

for any fixed choice of the p constants c_1, c_2, …, c_p, called the **scaling factors** of the linear combination. That is, $\mathbf{v} \in \mathbb{R}^n$ is a linear combination of the vectors from the set S if there exist scaling factors c_1, c_2, …, c_p, such that

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_p\mathbf{u_p}.$$

2. Consider $\mathbf{v} \in \mathbb{R}^n$ and let A be an $m \times n$ matrix. Assume now that $\mathbf{v}$ is a linear combination of the vectors from S with scaling factors $c_1, c_2, \ldots, c_p$. Then

$$A\mathbf{v} = c_1 A\mathbf{u_1} + c_2 A\mathbf{u_2} + \cdots c_p A\mathbf{u_p}.$$

3. Consider an $m \times n$ matrix A in the form

$$A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}],$$

where $\mathbf{a_j} \in \mathbb{R}^m$ for $j = 1, 2, \ldots, n$. Consider a vector $\mathbf{x} \in \mathbb{R}^n$ given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then the **matrix-vector product** $A\mathbf{x}$ is defined as the linear combination of the set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$ with scaling factors $x_1, x_2, \ldots, x_n$, i.e.

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}.$$

Remark: *See also* **Theoretical Remark 2.1 (3)** *where the matrix-vector product* $A\mathbf{x}$ *is discussed.*

Problem 3.1.1.

Consider the following set of five vectors in $\mathbb{R}^3$:

$$S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}, \mathbf{u_5}\},$$

where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$\mathbf{u_4} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{u_5} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

Consider also the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

a) Show that $\mathbf{v}$ is a linear combination of the vectors in the set S and give the linear combination explicitly.

b) Is $\mathbf{v}$ a linear combination of the set of vectors $\{\mathbf{u_1},\ \mathbf{u_2}\}$? Justify your answer.

c) Let A be an unspecified 3×3 matrix, such that

$$A\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad A\mathbf{u_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad A\mathbf{u_3} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

$$A\mathbf{u_4} = \begin{pmatrix} -8 \\ 11 \\ -18 \end{pmatrix}, \quad A\mathbf{u_5} = \begin{pmatrix} 22 \\ -13 \\ 32 \end{pmatrix}.$$

Find $A\mathbf{v}$ explicitly.

Solution 3.1.1.

a) We have to show that there exist real constants (scaling factors), c_1, c_2, c_3, c_4 and c_5, such that

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} + c_4\mathbf{u_4} + c_5\mathbf{u_5}.$$

We write this vector equation in the form of a matrix equation, namely

$$[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\ \mathbf{u_4}\ \mathbf{u_5}]\,\mathbf{c} = \mathbf{v},$$

where

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \in \mathbb{R}^5.$$

With the given vectors $\mathbf{u_j}$, we have

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 3 \\ 1 & 0 & 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

and the following corresponding augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 3 & 2 \\ 1 & 0 & 3 & 0 & 4 & 1 \end{pmatrix}.$$

Applying several elementary row operations on this augmented matrix, we obtain its unique reduced echelon form, namely

$$\begin{pmatrix} 1 & 0 & 0 & 9 & -14 & -5 \\ 0 & 1 & 0 & -4 & 9 & 4 \\ 0 & 0 & 1 & -3 & 6 & 2 \end{pmatrix}.$$

We conclude that the constants c_4 and c_5 can be chosen arbitrarily, so we let

$$c_4 = t, \qquad c_5 = s,$$

where t and s are arbitrary real parameters. From the above reduced echelon form we then have

$$\begin{aligned} c_1 &= -9t + 14s - 5 \\ c_2 &= 4t - 9s + 4 \\ c_3 &= 3t - 6s + 2, \end{aligned}$$

so that

$$\mathbf{c} = \begin{pmatrix} -9t + 14s - 5 \\ 4t - 9s + 4 \\ 3t - 6s + 2 \\ t \\ s \end{pmatrix}.$$

These are the scaling factors of the linear combination, so that

$$\mathbf{v} = (-9t + 14s - 5)\,\mathbf{u_1} + (4t - 9s + 4)\,\mathbf{u_2} + (3t - 6s + 2)\,\mathbf{u_3} + t\,\mathbf{u_4} + s\,\mathbf{u_5}$$

for all $t \in \mathbb{R}$ and all $s \in \mathbb{R}$. We can therefore choose $t = s = 0$ to find the simplest linear combination:

$$\mathbf{v} = -5\mathbf{u_1} + 4\mathbf{u_2} + 2\mathbf{u_3}.$$

b) We need to establish the existence of scaling factors c_1 and c_2, such that

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2}.$$

That is, we need to establish the consistency of the system

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

The associated augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

so that an echelon form becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

the last row of which indicates that the system is inconsistent, as it implies

$$c_1\,0 + c_2\,0 = 1.$$

We therefore conclude that there exist no constants c_1 and c_2 for which $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u_1}$ and $\mathbf{u_2}$.

c) In part a) abovewe have established the linear combination

$$\mathbf{v} = (-9t + 14s - 5)\,\mathbf{u_1} + (4t - 9s + 4)\,\mathbf{u_2} + (3t - 6s + 2)\,\mathbf{u_3} + t\,\mathbf{u_4} + s\,\mathbf{u_5}$$

$$\text{for all } t \in \mathbb{R} \text{ and all } s \in \mathbb{R}$$

and, by setting $t = s = 0$, the simplest linear combination

$$\mathbf{v} = -5\mathbf{u_1} + 4\mathbf{u_2} + 2\mathbf{u_3}.$$

Therefore

$$\begin{aligned} A\mathbf{v} &= A\,(-5\mathbf{u_1} + 4\mathbf{u_2} + 2\mathbf{u_3}) \\ &= -5A\mathbf{u_1} + 4A\mathbf{u_2} + 2A\mathbf{u_3} \\ &= -5\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix}. \end{aligned}$$

Of course we could, alternately, do the calculations using the combination with the arbitrary s and t parameters. This gives the same result:

$$\begin{aligned} A\mathbf{v} &= A(-9t + 14s - 5)\,\mathbf{u_1} + A(4t - 9s + 4)\,\mathbf{u_2} + A(3t - 6s + 2)\,\mathbf{u_3} + At\,\mathbf{u_4} + As\,\mathbf{u_5} \\ &= (-9t + 14s - 5)\,A\mathbf{u_1} + (4t - 9s + 4)\,A\mathbf{u_2} + (3t - 6s + 2)\,A\mathbf{u_3} + t\,A\mathbf{u_4} + s\,A\mathbf{u_5} \\ &= (-9t + 14s - 5)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + (4t - 9s + 4)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + (3t - 6s + 2)\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \\ &+t\begin{pmatrix} -8 \\ 11 \\ -18 \end{pmatrix} + s\begin{pmatrix} 22 \\ -13 \\ 32 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ -4 \\ 11 \end{pmatrix}. \end{aligned}$$

Problem 3.1.2.

Consider the vector

$$\mathbf{v} = \begin{pmatrix} k \\ 4 \\ 2 \\ 2 \end{pmatrix} \in \mathbb{R}^4$$

as well as the following set of vectors in $\mathbb{R}^4$:

$$S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\},$$

where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ k \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ k \\ 1 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ k \end{pmatrix}.$$

Here k is an unspecified real parameter. Determine all values of k, such that $\mathbf{v}$ is a linear combination of the vectors from the set S.

Solution 3.1.2.

Since $\mathbf{v}$ should be a linear combination of the vectors $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$, we have

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3}$$

or, in matrix form

$$\begin{pmatrix} k \\ 4 \\ 2 \\ 2 \end{pmatrix} = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \quad \text{That is}$$

$$\begin{pmatrix} k \\ 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

We now find all values of $k \in \mathbb{R}$, such that the above system is consistent, i.e. such that there exist real values for c_1, c_2 and c_3 that satisfy the system. The associated augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & k \\ k & 1 & 1 & 4 \\ 1 & k & 1 & 2 \\ 1 & 1 & k & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & k \\ 0 & 1-k & 1-k & 4-k^2 \\ 0 & k-1 & 0 & 2-k \\ 0 & 0 & k-1 & 2-k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & k \\ 0 & 1-k & 1-k & 4-k^2 \\ 0 & 0 & 1-k & -(k-2)(k+3) \\ 0 & 0 & 0 & -(k-2)(k+4). \end{pmatrix}.$$

From the last row of the above echelon form we have

$$c_1 0 + c_2 0 + c_3 0 = -(k-2)(k+4)$$

so that the system is consistent if and only if $k = 2$ or $k = -4$. Therefore, $\mathbf{v}$ is a linear combination of $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ if and only if $k = 2$ or $k = -4$.

3.2 Spanning sets of vectors

In this section we introduce the concept of a spanning set. That is, a finite set of vectors which span a subset of vectors in $\mathbb{R}^n$.

Theoretical Remarks 3.2.

Consider the set S of p vectors

$$S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}\},$$

where $\mathbf{u_j} \in \mathbb{R}^n$ for $j = 1, 2, \ldots, p$.

1. The set of **all** linear combinations of the vectors from S, denoted by span $\{S\}$, is a subset of $\mathbb{R}^n$, say W, that is said to be **spanned by** S. We write

$$W = \text{span}\,\{\mathbf{u_1},\ \mathbf{u_2},\ \ldots,\ \mathbf{u_p}\}, \quad \text{or simply} \quad W = \text{span}\,\{S\}.$$

   We say that S is the **spanning set of** W. Thus span $\{S\}$ consists of all linear combinations of vectors from S, i.e.

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_p\mathbf{u_p} \in W$$

   for every possible choice of the scaling factors $c_1,\ c_2,\ \ldots, c_p$. We write

$$W = \text{span}\,\{S\} = \{c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_p\mathbf{u_p} \text{ for all } c_1 \in \mathbb{R},\ c_2 \in \mathbb{R},\ \ldots,\ c_p \in \mathbb{R}\}.$$

2. In the sense of the above introduced spannig set, we can interpret the consistency of a linear system as follows:

   The linear system

$$A\mathbf{x} = \mathbf{b},$$

   is **consistent** if and only if

$$\mathbf{b} \in \text{span}\,\{\mathbf{a_1},\ \mathbf{a_2},\ \ldots,\ \mathbf{a_n}\},$$

   where A is an $m \times n$ matrix given by $A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}]$, $\mathbf{a_j} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$.

Problem 3.2.1.

Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} k \\ 6 \\ 2 \end{pmatrix},$$

where k is an unspecified real constant. Find all values of k, such that

a) $\mathbf{w} \in \text{span}\,\{\mathbf{u},\ \mathbf{v}\}$.

b) $\mathbf{u} \in \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$.

c) Is $\mathbf{0} \in \text{span}\,\{\mathbf{u},\ \mathbf{v}\}$, where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

d) Which of the following systems are consistent?

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}, \quad [\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}, \quad [\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

e) Find all values of k for which the system

$$[\mathbf{u}\ \mathbf{w}]\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is consistent.

Solution 3.2.1.

a) If vector $\mathbf{w}$ is an element span $\{\mathbf{u},\ \mathbf{v}\}$, then $\mathbf{w}$ must be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$. That is, there must exist scaling factors (real constants) c_1 and c_2, such that

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}.$$

Writing this as a matrix equation, we have

$$\begin{pmatrix} 1 & -1 \\ -2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} k \\ 6 \\ 2 \end{pmatrix}$$

so that the associated augmented matrix and one of its echelon forms are

$$\begin{pmatrix} 1 & -1 & k \\ -2 & 1 & 6 \\ 3 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & k \\ 0 & -1 & 2k+6 \\ 0 & 0 & 11k+44 \end{pmatrix}.$$

We conclude that the system is consistent if and only if $k = -4$.
Thereofore, $\mathbf{w} \in \text{span}\{\mathbf{u},\ \mathbf{v}\}$ if and only if $k = -4$.

b) If vector $\mathbf{u}$ is an element of span $\{\mathbf{v},\ \mathbf{w}\}$, then $\mathbf{u}$ must be a linear combination of the vectors $\mathbf{v}$ and $\mathbf{w}$. But we already know from part a) above, that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, which means that there exist scaling factors c_1 and c_2, such that

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}.$$

Therefore, we have

$$\mathbf{u} = -\frac{c_2}{c_1}\mathbf{v} + \frac{1}{c_1}\mathbf{w},$$

so that we can conclude that $\mathbf{u} \in \text{span}\{\mathbf{v},\ \mathbf{w}\}$ if and only if $k = -4$, i.e. the same value of k as in part a) above.

c) The zero-vector, $\mathbf{0} \in \mathbb{R}^3$, is always an element of any spanning set of $\mathbb{R}^3$, since $\mathbf{0}$ is always a linear combination with zero scaling factors of the vectors that span the set. In the current case, namely span $\{\mathbf{u},\ \mathbf{v}\}$, we have

$$\mathbf{0} = 0\mathbf{u} + 0\mathbf{v}.$$

d) Note that

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = x_1\mathbf{u} + x_2\mathbf{v}, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore the system

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} k \\ 6 \\ 2 \end{pmatrix}$$

is consistent if and only if $\begin{pmatrix} k \\ 6 \\ 2 \end{pmatrix} \in \text{span}\{\mathbf{u},\ \mathbf{v}\}$. In part a) above, we have already established that $\mathbf{w} \in \text{span}\{\mathbf{u},\ \mathbf{v}\}$ if and only if $k = -4$. Thus, the system

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

is inconsistent, while the system

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}$$

is consistent. Clearly the homogeneous system

$$[\mathbf{u}\ \mathbf{v}]\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is also consistent.

e) The homogeneous system

$$[\mathbf{v}\ \mathbf{w}]\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is consistent for all $k \in \mathbb{R}$, as

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is always a solution (the trivial- or zero-solution) of the system.

Problem 3.2.2.

Consider the following three vectors in $\mathbb{R}^4$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 7 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$$

and let W denote the set of vectors spanned by $\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$, i.e. let

$$W = \text{span}\,\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}.$$

Which of the following four vectors belong to W?

$$\mathbf{v_1} = \begin{pmatrix} -5 \\ -3 \\ 9 \\ 10 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution 3.2.2.

To answer whether $\mathbf{v_1} \in \operatorname{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ we need to establish whether there exist scaling factors c_1, c_2 and c_3, such that $\mathbf{v_1}$ is a linear combination of $\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}$, i.e.

$$\mathbf{v_1} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3}$$

The matrix equation is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 4 & -3 \\ -3 & 7 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 9 \\ 10 \end{pmatrix}$$

and the associated augmentd matrix is

$$\begin{pmatrix} 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & -3 \\ 2 & 4 & -3 & 9 \\ -3 & 7 & 1 & 10 \end{pmatrix}.$$

After applying several elementary row operations, we obtain the following reduced echelon form of the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

from which we conclude that $\mathbf{v_1} \in \operatorname{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$, where

$$\mathbf{v_1} = -2\mathbf{u_1} + \mathbf{u_2} - 3\mathbf{u_3}.$$

We follow the same procedure to establish whether $\mathbf{v_2} \in \operatorname{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$. This leads to the following augmented matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & -3 & 1 \\ -3 & 7 & 1 & 0 \end{pmatrix}$$

and the reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 19/28 \end{pmatrix}.$$

From the above reduced echelon form we conclude that $\mathbf{v_2}$ does not belong to the span $\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$, as $\mathbf{v_2}$ cannot be written as a linear combination of the vectors $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$.

Since $\mathbf{v_3} \in \mathbb{R}^3$ it cannot belong to a spanning set that is spanned by vectors in $\mathbb{R}^4$. For $\mathbf{v_4}$, we have

$$\mathbf{v_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\mathbf{u_1} + 0\mathbf{u_2} + 0\mathbf{u_3} \in \operatorname{span}\{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}.$$

3.3 Linearly dependent and independent sets of vectors

We introduce the concept of a linearly dependent sets and a linearly independent set of vectors in $\mathbb{R}^n$. We discuss the importance of linearly independet sets for a spanning set of vectors.

Theoretical Remarks 3.3.

Consider the set S of p vectors

$$S = \{\mathbf{u_1},\ \mathbf{u_2},\ \ldots,\ \mathbf{u_p}\},$$

where $\mathbf{u_j} \in \mathbb{R}^n$ for every $j = 1, 2, \ldots, p$.

The set S is a **linearly independent set** in $\mathbb{R}^n$ if the vector equation

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_p\mathbf{u_p} = \mathbf{0}$$

can only be satisfied if all scaling factors are zero, i.e. $c_1 = 0,\ c_2 = 0,\ \ldots,\ c_p = 0$. If there exists any non-zero scaling factors for which the above vector equation is satisfied, then the set S is a **linearly dependent set**.

Remark: *Consider a set of n vectors*

$$S = \{\mathbf{u_1},\ \mathbf{u_2},\ \ldots,\ \mathbf{u_n}\},$$

where $\mathbf{u_j} \in \mathbb{R}^n$ for every $j = 1, 2, \ldots, n$. If the set S is linearly independent and the set S spans $\mathbb{R}^n$, then S is a **basis** *for $\mathbb{R}^n$ and we say that the* **dimension** *of $\mathbb{R}^n$ is n. The* **standard basis**, *$\{\mathbf{e_1},\ \mathbf{e_2},\ \ldots, \mathbf{e_n}\}$ is an example of a basis for $\mathbb{R}^n$. These concepts of basis and dimension is discussed in detail in* **Part 2** *of this series, subtitled* **General Vector Spaces**.

For linearly independent and linearly dependent sets we have the following

Properties:

a) Assume that S is a linearly independent set. Then **all** subsets of vectors from S are also linearly independent sets in $\mathbb{R}^n$.

b) Assume that S is a linearly dependent set. Then there may exist subsets of two or more vectors from S which are linearly independent sets in $\mathbb{R}^n$.

c) Let S be a set that consist of p vectors in $\mathbb{R}^n$. If $p > n$ then S is a linearly dependent set.

d) Consider a set Q of n vectors in $\mathbb{R}^n$, namely

$$Q = \{\mathbf{a_1},\ \mathbf{a_2},\ \ldots,\ \mathbf{a_n}\},$$

where $\mathbf{a_j} \in \mathbb{R}^n$ for all $j = 1, 2, \ldots, n$ and consider the $n \times n$ matrix

$$A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}].$$

We have the following properties:

i) The set Q is linearly independent if and only if the reduced echelon form of A is the identity matrix I_n, i.e. A and I_n are row-equivalent

$$A \sim I_n.$$

ii) The set Q is linearly independent if and only if the determinant of A is non-zero, i.e.

$$\det A \neq 0,$$

so that A is an invertible matrix.

iii) The set Q is linearly independent if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{R}^n$ for all $\mathbf{b} \in \mathbb{R}^n$.

e) Consider the set Q_2 of two non-zero vectors in $\mathbb{R}^3$, namely

$$Q_2 = \{\mathbf{u_1},\ \mathbf{u_2}\}.$$

Then span $\{Q_2\}$ will span a plane Π in $\mathbb{R}^3$ that contains the origin $(0,0,0)$, if and only if Q_2 is a linearly independent set. That is, every vector in the plane Π is a linear combination of the vectors $\mathbf{u_1}$ and $\mathbf{u_2}$.

f) Consider the set Q_3 of three non-zero vectors in $\mathbb{R}^3$, namely

$$Q_3 = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}.$$

Then span $\{Q_3\}$ spans $\mathbb{R}^3$ if and only if Q_3 is a linearly independent set. That is, every vector in $\mathbb{R}^3$ is a linear combination of the vectors $\mathbf{u_1},\ \mathbf{u_2}$ and $\mathbf{u_3}$. Moreover, if Q_3 is a linearly dependent set with exactly two linearly independent vectors, then span $\{Q_3\}$ spans a plane through $(0,0,0)$ (of course the same is true for any finite set of vectors which contains a subset of exactly two linearly independent vectors). Note that a line ℓ through $(0,0,0)$ can be spanned by any non-zero vector with coordinates on ℓ.

Problem 3.3.1.

Consider the following two vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} k \\ -4 \\ 2 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all real values of k, such that the set $S = \{\mathbf{u},\ \mathbf{v}\}$ is a linearly independent set, as well as all the real values of k for which S is a linearly dependent set.

b) Find all real values of k, such that $\mathbf{u}$ and $\mathbf{v}$ span a plane in $\mathbb{R}^3$ and give the equation of that plane.

c) Find all real values of k, such that $\mathbf{u}$ and $\mathbf{v}$ span a line in $\mathbb{R}^3$ and give the equation of that line in parametric form.

Solution 3.3.1.

a) To establish whether the set $S = \{\mathbf{u},\ \mathbf{v}\}$ is a linearly independent set (or a linearly dependent set), we consider the vector equation

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}.$$

In matrix form this becomes

$$\begin{pmatrix} 1 & k \\ 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the system is homogeneous, we only have to look at the coefficient matrix to establish the consistency of the system. We have

$$\begin{pmatrix} 1 & k \\ 2 & -4 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & k \\ 0 & -2k-4 \\ 0 & k+2 \end{pmatrix} \sim \begin{pmatrix} 1 & k \\ 0 & k+2 \\ 0 & 0 \end{pmatrix}.$$

The reduced system is therefore

$$\begin{aligned} &c_1 + c_2 k = 0 \\ &c_2(k+2) = 0. \end{aligned}$$

Therefore the system has the trivial solution, i.e. $c_1 = 0$ and $c_2 = 0$, if and only if $k \in \mathbb{R}\backslash\{-2\}$. We conclude that the set S is linearly independent for $k \in \mathbb{R}\backslash\{-2\}$ and linearly dependent for $k = -2$.

b) The set of two vectors, $S = \{\mathbf{u},\ \mathbf{v}\}$, will span a plane in $\mathbb{R}^3$ if and only if S is a linearly independent set, i.e. for all value $k \in \mathbb{R}\backslash\{-2\}$, as established in part a) above. This plane contains the origin $(0, 0, 0)$ as well as all those vectors that are linear combinations of $\mathbf{u}$ and $\mathbf{v}$ for all $k \in \mathbb{R}\backslash\{-2\}$. To find the equation of the plane that is spanned by S, we first calculate the normal vector $\mathbf{n}$ for the plane by the use of the cross-product (see **Theoretical Remark 1.2** and **Theoretical Remark 1.3**). We have

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & 2 & -1 \\ k & -4 & 2 \end{vmatrix} = -(k+2)\mathbf{e_2} - (2k+4)\mathbf{e_3}.$$

Then we calculate the dot product of $\mathbf{n}$ with an arbitrary point on the plane, say the point (x, y, z), which must be zero as long as (x, y, z) is on the plane. Thus

$$\mathbf{n} \cdot (x, y, z) = (0,\ -k-2,\ -2k-4) \cdot (x, y, z) = -(k+2)y - (2k+4)z = 0.$$

Therefore, the equation of the plane that is spanned by S is

$$-(k+2)y - (2k+4)z = 0 \quad \text{for all } k \in \mathbb{R}\backslash\{-2\}.$$

Note that the equation of the plane depends on k.

c) The set of two vectors in the set $S = \{\mathbf{u},\ \mathbf{v}\}$ will span a line ℓ in $\mathbb{R}^3$ if and only if S is a linearly dependent set, i.e. for the value $k = -2$, as established in part a) above. Those are the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix},$$

as well as all linear combinations of $\mathbf{u}$ and $\mathbf{v}$, as all these vectors are on the line ℓ. Obviously the line ℓ passes through the origin $(0, 0, 0)$. To find the equation of ℓ that is spanned by S, we just have to multiply any vector on ℓ with an arbitrary parameter, t, say the vector $\mathbf{u}$. Thus a parametric equation of the line is

$$\ell : \begin{cases} x = t \\ y = 2t \\ z = -1t \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$

Problem 3.3.2.

Consider the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ with

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that the set S is a linearly independent set and find also all values of k such that the set is linearly dependent.

b) Consider $W = \text{span}\,\{\mathbf{u_1},\ \mathbf{u_2}\}$. Find all values of k, such that $\mathbf{u_3} \in W$.

c) Find all values of k, such that $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ span a plane in $\mathbb{R}^3$ and give the equation of that plane explicitly.

d) Find all values of k, such that $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ span $\mathbb{R}^3$.

e) Do there exist values for k, such that $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ span a line in $\mathbb{R}^3$?

f) Consider the matrix $A = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}]$ and find all values of k, such that A is row equivalent to I_3 (the 3×3 identity matrix) and find also all values of k for which A is invertible.

g) Find all values of k, such that the system

$$[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}]\mathbf{x} = \mathbf{b}$$

has a unique solution.

Solution 3.3.2.

a) To establish whether the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ is linearly independent or linearly dependent, we have to consider the vector equation

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} = \mathbf{0}.$$

In matrix form we have

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & k \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & k-2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & k-8 \end{pmatrix}.$$

Thus there exist non-zero solutions for c_1, c_2 and c_3 if and only if $k-8=0$. Therefore the set S is linearly independent for all values $k \in \mathbb{R}\backslash\{8\}$ and linearly dependent for $k = 8$.
Since the coefficient matrix is a square matrix, we may also establish the linear independence of the set by calculating the determinant of the system's coefficient matrix; let's name this matrix A. We have $\det A = k - 8$. The columns of A are linearly independent if and only if $\det A \neq 0$. Hence we have the same conclusion as above.

b) In order to determine whether $\mathbf{u_3} \in \text{span}\{\mathbf{u_1},\ \mathbf{u_2}\} = W$, we need to investigate the consistency of the non-homogeneous system $c_1\mathbf{u_1} + c_2\mathbf{u_2} = \mathbf{u_3}$. Clearly this system can only be consistent if the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ is linearly dependent and then $\mathbf{u_3} \in W$. We have already established in part a) above, that the set S is linearly dependent for $k = 8$. Hence $\mathbf{u_3} \in W$ for $k = 8$.

c) In order to span a plane in $\mathbb{R}^3$ we need exactly two linearly independent vectors. First, we note that the set $\{\mathbf{u_1},\ \mathbf{u_2}\}$ is clearly a linearly independent set, since

$$\mathbf{u_1} \neq \alpha\mathbf{u_2} \quad \text{for all } \alpha \in \mathbb{R}.$$

So to span a plane Π in $\mathbb{R}^3$ with all three vectors in the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$, we need to make sure that the set S is a linearly dependent set with a linearly independent subset of two vectors. We have establsihed in part a), that S is a linearly dependent set for $k = 8$. Hence

$$\Pi = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}\}.$$

Note that the same plane Π can also be spanned by any subset of two linearly independent vectors from the set S. We have

$$\Pi = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \right\}.$$

To find the equation of the plane Π we can use any of the above given spanning sets. We'll use $\Pi = \text{span}\{\mathbf{u_1}, \ \mathbf{u_2}\}$. We calculate the normal vector $\mathbf{n}$ of the plane and then calculate the dot product with an arbitrary point (x, y, z) on the plane. (see **Theoretical Remark 1.2** and **Theoretical Remark 1.3**). We have

$$\mathbf{n} = \mathbf{u_1} \times \mathbf{u_2} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -\mathbf{e_1} - 2\mathbf{e_2} + \mathbf{e_3} = (-1, -2, 1).$$

Then

$$\mathbf{n} \cdot (x, y, z) = (-1, -2, 1) \cdot (x, y, z) = -x - 2y + z = 0.$$

Hence the equation of the plane Π is

$$-x - 2y + z = 0.$$

d) In order to span $\mathbb{R}^3$ we need a set S of three vectors in $\mathbb{R}^3$ such that every vector in $\mathbb{R}^3$ can be written as a linear combination of those three vectors. For this to be possible, the set S needs to be linearly independent. Let $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$. In part a) we have already established that S is a linearly independent set for all $k \in \mathbb{R}\backslash\{8\}$. Thus

$$\mathbb{R}^3 = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\k\end{pmatrix}\right\} \quad \text{for all } k \in \mathbb{R}\backslash\{8\}.$$

e) We can not span a line in $\mathbb{R}^3$ by using all three vectors in the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$, since the subset $\{\mathbf{u_1},\ \mathbf{u_2}\}$ is already linearly independent (so those span a plane as shown in part c) above).

f) The matrix $A = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}]$ is row equivalent to I_3 if and only if $\det A \neq 0$ and thus A is invertible. We have

$$\det A = \begin{vmatrix}1 & 0 & 2\\0 & 1 & 3\\1 & 2 & k\end{vmatrix} = k - 8.$$

Thus $A \sim I_3$ if and only if $k \in \mathbb{R}\backslash\{8\}$. Moreover A^{-1} exists if and only if $k \in \mathbb{R}\backslash\{8\}$.

g) The system $A\mathbf{x} = \mathbf{b}$, with

$$A = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}],$$

has a unique solution for all $\mathbf{b} \in \mathbb{R}^3$ if and only if A is an invertible matrix. That is, the system has a unique solution if and only if $\det A \neq 0$. In part f) we have established that this is the case for all $k \in \mathbb{R}\backslash\{8\}$. Hence, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $k \in \mathbb{R}\backslash\{8\}$.

Problem 3.3.3.

Consider the set of vectors $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\}$ in $\mathbb{R}^4$ with

$$\mathbf{u_1} = \begin{pmatrix}1\\1\\-2\\-3\end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix}-1\\-9\\k\\11\end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix}-1\\3\\1\\-1\end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix}k\\-10\\-4\\6\end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that S is a linearly independent set.

b) Find all values of k, for which S is a linearly dependent set and list all possible subsets of three linearly independent vectors in S with their corresponding k values.

Solution 3.3.3.

a) To establish the linear independence of the set S, we consider the matrix A that contains the vectors in the set S as column entries:

$$A = \begin{pmatrix} 1 & -1 & -1 & k \\ 1 & -9 & 3 & -10 \\ -2 & k & 1 & -4 \\ -3 & 11 & -1 & 6 \end{pmatrix}.$$

The columns of A are linearly independent if and only if $\det A \neq 0$. That is, the set S is linearly independent if and only if $\det A \neq 0$. We obtain

$$\det A = -8k^2 + 48k - 64 = -8(k-2)(k-4).$$

Hence, S is a linearly independent set for all $k \in \mathbb{R}\backslash\{2,\ 4\}$.

b) By a) above, we know that S is a linearly dependent set for both $k = 2$ as well as for $k = 4$. To establish a subset of three linearly independent vectors, we consider $k = 2$ and $k = 4$ in two separate cases:
<u>Let $k = 2$.</u> Then

$$A \sim \begin{pmatrix} 1 & -1 & -1 & 2 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the above it is clear that there exist two subsets that contain three linearly independent vectors in the set S with $k = 2$, namely the subsets

$$S_1 = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\} \quad \textit{or} \quad S_2 = \{\mathbf{u_1},\ \mathbf{u_3},\ \mathbf{u_4}\},$$

where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -1 \\ -9 \\ 2 \\ 11 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix} 2 \\ -10 \\ -4 \\ 6 \end{pmatrix}.$$

<u>Let $k = 4$.</u> Then

$$A \sim \begin{pmatrix} 1 & -1 & -1 & 4 \\ 0 & 4 & -2 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the above it follows that there exist also two subsets that contain three linearly independent vectors in the set S with $k = 4$, namely the subsets

$$S_3 = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_4}\} \quad or \quad S_4 = \{\mathbf{u_1},\ \mathbf{u_3},\ \mathbf{u_4}\},$$

where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -1 \\ -9 \\ 4 \\ 11 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix} 4 \\ -10 \\ -4 \\ 6 \end{pmatrix}.$$

3.4 Exercises

1. Consider the following two vectors in $\mathbb{R}^2$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ k \end{pmatrix}, \qquad \mathbf{u_2} = \begin{pmatrix} k \\ k+2 \end{pmatrix},$$

where k is an unspecified real parameter. Find all values of k, such that $S = \{\mathbf{u_1},\ \mathbf{u_2}\}$ is a linearly independent set and all values of k, such that S is a linearly dependent set.

[**Answer:** S is a linearly independent set for all $k \in \mathbb{R}\backslash\{-1,\ 2\}$ and S is a linearly dependent set for $k = -1$ or $k = 2$.]

2. Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}, \qquad \mathbf{u_3} = \begin{pmatrix} k \\ 1 \\ 3 \end{pmatrix},$$

where k is an unspecified real parameter. Find all values of k, such that $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ is a linearly independent set and all values of k, such that S is a linearly dependent set.

[**Answer:** S is a linearly independent set for all $k \in \mathbb{R}$ and S can therefore not be a linearly dependent set for any k.]

3. Consider the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\}$ with the following vectors in $\mathbb{R}^4$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ -2 \\ -3 \\ 3 \end{pmatrix}, \qquad \mathbf{u_2} = \begin{pmatrix} -3 \\ k \\ 9 \\ -9 \end{pmatrix}, \qquad \mathbf{u_3} = \begin{pmatrix} -4 \\ 6 \\ k \\ -4 \end{pmatrix}, \qquad \mathbf{u_4} = \begin{pmatrix} -1 \\ -1 \\ -6 \\ 9 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that S is a linearly independent set.

[**Answer:** S is a linearly independent set for all $k \in \mathbb{R}\backslash\{6\}$.]

b) Give all possible linearly independet subsets of S.

[**Answer:** For $k \in \mathbb{R}\backslash\{6\}$ every subset of S is linearly independent (since S is a linearly independent set in this case).
For $k = 6$ the set S is linearly dependent and it has four linearly independent subsets consisting of two vectors each, namely

$$S_1 = \{\mathbf{u_1},\ \mathbf{u_3}\}, \quad S_2 = \{\mathbf{u_1},\ \mathbf{u_4}\}, \quad S_3 = \{\mathbf{u_2},\ \mathbf{u_3}\}, \quad S_4 = \{\mathbf{u_2},\ \mathbf{u_4}\},$$

$$S_5 = \{\mathbf{u_3},\ \mathbf{u_4}\}.\ ]$$

4. Consider the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\}$ with the following vectors in $\mathbb{R}^4$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 5 \\ -3 \\ -5 \\ 15 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 4 \\ 1 \\ k \\ 12 \end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix} k \\ -1 \\ 4 \\ -12 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that S is a linearly independent set and also all values of k, such that S is a linearly dependent set.

[**Answer:** S is a linearly independent set for all $k \in \mathbb{R}\backslash\{-4\}$ and a linearly dependent set for $k = -4$.]

b) Give all possible linearly independet subsets of S.

[**Answer:** For $k \in \mathbb{R}\backslash\{-4\}$ every subset of S is linearly independent (since S is a linearly independent set in this case).
For $k = -4$ the set S has five subsets that are linearly independent, namely

$$S_1 = \{\mathbf{u_1},\ \mathbf{u_2}\}, \quad S_2 = \{\mathbf{u_1},\ \mathbf{u_3}\}, \quad S_3 = \{\mathbf{u_1},\ \mathbf{u_4}\}, \quad S_4 = \{\mathbf{u_2},\ \mathbf{u_3}\},$$

$$S_5 = \{\mathbf{u_2},\ \mathbf{u_4}\}.\ ]$$

5. Consider the set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ with the following vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} k \\ 0 \\ 1 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that S is a linearly independent set.

[**Answer:** All $k \in \mathbb{R}\backslash\{-1\}$.]

b) Find all values of k, such that the vectors of the set S span $\mathbb{R}^3$, i.e.

$$\mathbb{R}^3 = \text{span}\,\{S\}.$$

[**Answer:** All $k \in \mathbb{R}\backslash\{-1\}$.]

6. Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where $a,\ b$ and c are real parameters. Consider now the set

$$S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$$

a) Find all real values of the parameters $a,\ b$ and c, such that S is a linearly dependent set in $\mathbb{R}^3$.

[**Answer:** S is a linearly dependent set for all $a,\ b,\ c \in \mathbb{R}$.]

b) Find all values of the parameters $a,\ b$ and c, such that S spans a line ℓ in $\mathbb{R}^3$ and give this line explicitly in parametric form in terms of one parameter.

[**Answer:** S will span a line ℓ in $\mathbb{R}^3$ if and only if $\mathbf{u_3} = t\mathbf{u_1}$, i.e.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ -t \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.$$

Then a parametric equation for ℓ takes the form

$$\ell : \begin{cases} x = 2t \\ y = t \\ z = -t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$]

7. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} k \\ -3 \\ 3 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix},$$

where k is an unspecified real parameter. Consider also the set

$$S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}.$$

a) Find all real value of k, such that the set S spans a plane W in $\mathbb{R}^3$. That is, find all $k \in \mathbb{R}$, such that

$$W = \text{span}\{S\}.$$

Give the equation of the plane W.

[**Answer:** The vectors of the set S span a plane W for all $k \in \mathbb{R}\backslash\{-6\}$ and the equation of the plane W is $y + z = 0$.]

b) For which value(s) of k is the system

$$[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}]\,\mathbf{x} = \mathbf{b}$$

consistent? Does $Q = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ span a plane in $\mathbb{R}^3$ for any of your obtained k values and, if so, is $\mathbf{b}$ a vector in this plane?

[**Answer:** The system is consistent for all $k \in \mathbb{R}\backslash\{-6\}$. The vectors in the set Q span a plane in $\mathbb{R}^3$ for all $k \in \mathbb{R}\backslash\{-6\}$. Moreover $\mathbf{b}$ is a vector in this plane.]

c) For which value(s) of k is the system

$$[\mathbf{u_1}\ \mathbf{u_3}]\,\mathbf{x} = \mathbf{c}$$

consistent? Does $Q = \{\mathbf{u_1},\ \mathbf{u_3}\}$ span a plane in $\mathbb{R}^3$ for any of your obtained k values and, if so, is $\mathbf{c}$ a vector in this plane?

[**Answer:** The system is inconsistent for all $k \in \mathbb{R}$. The vectors in the set Q span a plane in $\mathbb{R}^3$ for all $k \in \mathbb{R}\backslash\{-6\}$. Note: $\mathbf{c}$ is not a vector in this plane.]

d) For which value(s) of k is the system

$$[\mathbf{u_1}\ \mathbf{u_2}\ \ \mathbf{u_3}]\,\mathbf{x} = \mathbf{c}$$

consistent? Does $Q = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ span a plane in $\mathbb{R}^3$ for any of your k values and, if so, is $\mathbf{c}$ a vector in this plane?

[**Answer:** The system is inconsistent for all $k \in \mathbb{R}$. The vectors in the set Q span a plane in $\mathbb{R}^3$ for all $k \in \mathbb{R}\backslash\{-6\}$. Note: $\mathbf{c}$ is not a vector in this plane.]

e) Is the system

$$[\mathbf{u_1}\ \mathbf{u_2}]\,\mathbf{x} = \mathbf{c}$$

consistent?

[**Answer:** The system is inconsistent.]

8. Consider a set $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3}\}$ with the following vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 3 \\ k \\ 7 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values for k, such that S spans $\mathbb{R}^3$, i.e. find all values for k, such that

$$\mathbb{R}^3 = \text{span}\,\{S\}.$$

[**Answer:** For all $k \in \mathbb{R}\backslash\{12\}$.]

b) Find all values for k, such that S spans a plane Π in $\mathbb{R}^3$ and find the equation of that plane.

[**Answer:** For $k = 12$ the plane Π is $6x - 5y + 6z = 0$.]

c) Find all values for α, such that the vector

$$\mathbf{v} = \begin{pmatrix} 8 \\ 6 \\ \alpha \end{pmatrix}$$

is in the plane Π spanned by the vectors in the set S.

[**Answer:** $\mathbf{v} \in \Pi :\ 6x - 5y + 6z = 0$ if and only if $\alpha = -3$.]

9. Consider the set of vectors $S = \{\mathbf{u_1},\ \mathbf{u_2},\ \mathbf{u_3},\ \mathbf{u_4}\}$ in $\mathbb{R}^4$ with

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -1 \\ -9 \\ k \\ 11 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} -1 \\ 3 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix} k \\ -10 \\ -4 \\ 6 \end{pmatrix},$$

where k is an unspecified real parameter.

a) Find all values of k, such that S is a linearly independent set.

[**Answer:** S is a linearly independent set for all $k \in \mathbb{R}\backslash\{2\}$.]

b) Find all values of k, for which S is linearly dependent and list all possible subsets of two linearly independent vectors in S with their corresponding k values.

[**Answer:** S is a linearly dependent set for $k = 2$ and for this value of k there exists three linearly independent subsets containing two vectors, namely

$$S_1 = \{\mathbf{u_1},\ \mathbf{u_2}\}, \quad S_2 = \{\mathbf{u_1},\ \mathbf{u_3}\}, \quad S_3 = \{\mathbf{u_1},\ \mathbf{u_4}\}.$$]

c) Does there exist values of k for which there exists a subset of vectors in S that contains three linearly independent vectors?

[**Answer:** No.]

Chapter 4

Linear Transformations in Euclidean spaces

The aim of this chapter:

We treat *linear transformations* that act between Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ and describe the relation of such transformations to systems of linear equations. We introduce the so-called *standard matrix*, which gives a unique and complete description of linear transformations. We discuss many examples of linear transformations, we show how to derive their standard matrices and how to compose linear transformations. We introduce *injective transformations* and *surjective transformations*, and invesitgate invertible linear transformations that map vectors in the same Euclidean space.

4.1 Linear transformations: domain and range

In this section we address linear transformations and give several examples, where we also discuss the domain and the range of such transformations.

Theoretical Remarks 4.1.

Consider a **transformation** (or **mapping**) T that map a subset $\mathcal{D}_T$ of vectors from $\mathbb{R}^n$, called the **domain** of T, to vectors in $\mathbb{R}^m$. This is denoted by

$$T\colon \mathcal{D}_T \subseteq \mathbb{R}^n \to \mathbb{R}^m.$$

Let $\mathbf{x} \in \mathcal{D}_T$. Then we write

$$T\colon \mathbf{x} \mapsto T(\mathbf{x}) \in \mathbb{R}^m,$$

where $T(\mathbf{x})$ is known as the **image** of $\mathbf{x}$ under T.

1. The **co-domain** of $T\colon \mathcal{D}_T \subseteq \mathbb{R}^n \to \mathbb{R}^m$, denoted by C_T, is the Euclidean space $\mathbb{R}^m$. See Figure 4.1.

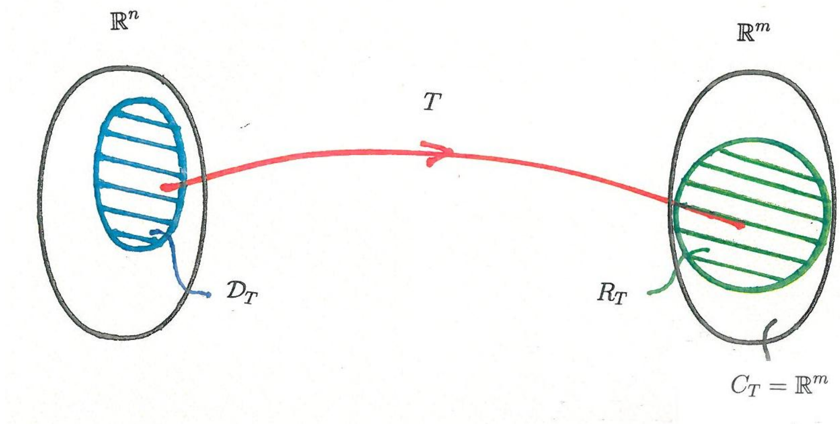

Figure 4.1: The domain, co-domain and range of a transformation $T\colon \mathcal{D}_T \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

2. The **range** of the transformation $T\colon \mathcal{D}_T \subseteq \mathbb{R}^n \to \mathbb{R}^m$, denoted by R_T, consists of a subset of vectors in $\mathbb{R}^m$, denoted by R_T, namely all those vectors in the co-domain $\mathbb{R}^m$ that are the images of all vectors $\mathbf{x}$ in $\mathcal{D}_T$. Hence $R_T \subseteq \mathbb{R}^m$. See Figure 4.1.

3. A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is known as a **linear transformation** with the domain $\mathbb{R}^n$ if it satisfies the following two conditions:

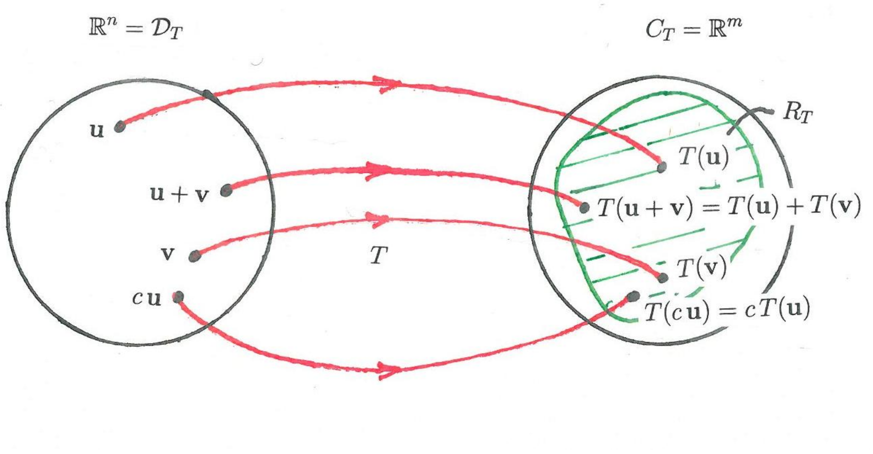

Figure 4.2: A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u} \in \mathbb{R}^n$ and all $\mathbf{u} \in \mathbb{R}^n$ (see Figure 4.2);

b) $T(c\mathbf{u}) = c\,T(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$ (see Figure 4.2).

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with domain $\mathbb{R}^n$. Then we have the following

Properties:

a) The zero-vector $\mathbf{0}_n$ of $\mathbb{R}^n$ is mapped to the zero-vector $\mathbf{0}_m$ of $\mathbb{R}^m$. That is $T(\mathbf{0}_n) = \mathbf{0}_m$.

b) $T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1\,T(\mathbf{u}) + c_2\,T(\mathbf{v})$ for all $\mathbf{u} \in \mathbb{R}^n$, all $\mathbf{v} \in \mathbb{R}^n$, all $c_1 \in \mathbb{R}$ and all $c_2 \in \mathbb{R}$.

Problem 4.1.1.

Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$, such that

$$T\colon (x_1,\ x_2) \mapsto (x_1 + x_2,\ 3x_1 + x_2,\ x_1 - x_2) \quad \text{for all } x_1,\ x_2 \in \mathbb{R}.$$

a) Show that T is a linear transformation.

b) What is the domain, the co-domain and the range of T.

c) Find $T(1, -2)$.

Solution 4.1.1.

a) Consider two arbitrary vectors in $\mathbb{R}^2$, say

$$\mathbf{x} = (x_1,\ x_2) \quad \text{and} \quad \mathbf{y} = (y_1,\ y_2).$$

To establish whether T is linear, we need to show that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\,\mathbf{x}) = c\,T(\mathbf{x})$ holds for all $c \in \mathbb{R}$. For the first condition we have

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= (x_1 + y_1 + x_2 + y_2,\ 3(x_1 + y_1) + x_2 + y_2,\ x_1 + y_1 - (x_2 + y_2)) \\ &= (x_1 + x_2 + y_1 + y_2,\ 3x_1 + x_2 + 3y_1 + y_2,\ x_1 - x_2 + y_1 - y_2) \end{aligned}$$

Moreover, we have

$$\begin{aligned} T(\mathbf{x}) + T(\mathbf{y}) &= (x_1 + x_2,\ 3x_1 + x_2,\ x_1 - x_2) + (y_1 + y_2,\ 3y_1 + y_2,\ y_1 - y_2) \\ &= (x_1 + x_2 + y_1 + y_2,\ 3x_1 + x_2 + 3y_1 + y_2,\ x_1 - x_2 + y_1 - y_2) \\ &= T(\mathbf{x}+\mathbf{y}). \end{aligned}$$

For the second condition we have

$$\begin{aligned} T(c\,\mathbf{x}) &= (c\,x_1 + c\,x_2,\ 3c\,x_1 + c\,x_2,\ c\,x_1 - c\,x_2) \\ &= c\,(x_1 + x_2,\ 3x_1 + x_2,\ x_1 - x_2) \\ &= c\,T(\mathbf{x}) \quad \text{for all } c \in \mathbb{R}. \end{aligned}$$

We conclude that T is a linear transformation.

b) The domain $\mathcal{D}_T$ is obviously R^2 as we allow all vectors $(x_1,\ x_2)$ to be mapped by T. The co-domain of T is $\mathbb{R}^3$, as vectors are being mapped from $\mathbb{R}^2$ to $\mathbb{R}^3$. To establish the range of T we need to find all the images in $\mathbb{R}^3$ of $(x_1,\ x_2)$ under T. For this, it is more convenient to write the transformation in matrix equation form. We note that, for every x_1 and x_2, the linear transformation $T\colon (x_1,\ x_2) \mapsto (b_1,\ b_2,\ b_3)$ maps as follows:

$$\begin{aligned} x_1 + x_2 &= b_1 \\ 3x_1 + x_2 &= b_2 \\ x_1 - x_2 &= b_3. \end{aligned}$$

Hence we have the matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

The given linear transformation T can therefore be written as follows:

$$T\colon \mathbf{x} \mapsto A\mathbf{x} = \mathbf{b} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

So to find the range of T, we need to find all $\mathbf{b} \in \mathbb{R}^3$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent. The associated augmented matrix and some of its row equivalent matrices are

$$\begin{pmatrix} 1 & 1 & b_1 \\ 3 & 1 & b_2 \\ 1 & -1 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - b_1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & 2b_1 - b_2 + b_3 \end{pmatrix}.$$

From the above echelon matrix we conclude that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if

$$2b_1 - b_2 + b_3 = 0.$$

We let $b_2 = t$ and $b_3 = s$, where t and s are arbitrary real parameters. Then we have

$$\mathbf{b} = \begin{pmatrix} t/2 - s/2 \\ t \\ s \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \quad \text{for all } t,\ s \in \mathbb{R}.$$

Hence we conclude that all the vectors in $\mathbb{R}^3$ that are images of $\mathbf{x}$ under T belong to the spanning set span $\{\mathbf{v_1},\ \mathbf{v_2}\}$, where

$$\mathbf{v_1} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v_2} = \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

so that the range R_T of T is a subset of $\mathbb{R}^3$, given by

$$R_T = \text{span}\,\{\mathbf{v_1},\ \mathbf{v_2}\}.$$

c) We find $T(1,\ -2)$:

$$T\colon (1,\ -2) \mapsto A\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \in R_T.$$

Problem 4.1.2.

Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, such that

$$T\colon (x_1,\ x_2,\ x_3) \mapsto (3x_1,\ 2x_2 + x_3 + 1) \quad \text{for all } x_1,\ x_2,\ x_3 \in \mathbb{R}.$$

Is T a linear transformation? Explain your answer.

Solution 4.1.2.

We investigate $T(\mathbf{x} + \mathbf{y})$, where

$$\mathbf{x} = (x_1,\ x_2,\ x_3), \quad \mathbf{y} = (y_1,\ y_2,\ y_3).$$

We have

$$T(\mathbf{x} + \mathbf{y}) = (3(x_1 + y_1),\ 2(x_2 + y_2) + x_3 + y_3 + 1),$$

and

$$\begin{aligned} T(\mathbf{x}) + T(\mathbf{y}) &= (3x_1,\ 2x_2 + x_3 + 1) + (3y_1,\ 2y_2 + y_3 + 1) \\ &= (3(x_1 + y_1),\ 2(x_2 + y_2) + x_3 + y_3 + 2). \end{aligned}$$

Thus $T(\mathbf{x} + \mathbf{y}) \neq T(\mathbf{x}) + T(\mathbf{y})$, so that T is not a linear transformation.

Problem 4.1.3.

Consider the transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, such that

$$T\colon \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

where A is any $m \times n$ matrix. Show that T is a linear transformation.

Solution 4.1.3.

Consider $T\colon \mathbb{R}^n \to \mathbb{R}^m$, such that

$$T\colon \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

where A is an $m \times n$ matrix. We show that T is a linear transformation for any given $m \times n$ matrix. Consider any two vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$. Then

$$\begin{aligned} T(\mathbf{x} + \mathbf{y}) &= A(\mathbf{x} + \mathbf{y}) \\ &= A\mathbf{x} + A\mathbf{y} \\ &= T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

Also

$$\begin{aligned} T(c\,\mathbf{x}) &= A(c\,\mathbf{x}) \\ &= c\,(A\mathbf{x}) \\ &= c\,T(\mathbf{x}) \quad \text{for all } c \in \mathbb{R}. \end{aligned}$$

We conclude that T is a linear transformation for any $m \times n$ matrix A.

4.2 Standard matrices and composite transformations

In this section we show how to find the standard matrix for a given linear transformation T. The standard matrix, which can be derived in terms of the standard basis vectors of the domain of T, gives a unique description of T. We also discuss linear composite tansformations, which result when several linear transformations are composed.

Theoretical Remarks 4.2.

1. **The Standard matrix of T:**

   Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation that map all vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. Then there exists a unique $m \times n$ matrix A, such that

   $$T\colon \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x} \in \mathbb{R}^m$$

   for every $\mathbf{x} \in \mathbb{R}^n$. This matix A is known as the **standard matrix** of T. In particular,

   $$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ \cdots\ T(\mathbf{e_n})],$$

   where $\{\mathbf{e_1},\ \mathbf{e_2},\ \cdots,\ \mathbf{e_n}\}$ is the **standard basis** of $\mathbb{R}^n$ with

   $$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

   Note: *The above derivation for A stems from the fact that every vector $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n) \in \mathbb{R}^n$, can uniquely be written as a linear combination of the standard basis vectors as follows:*

   $$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}.$$

2. Consider two linear transformations, T_1 and T_2, such that

   $$T_1\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad T_2\colon \mathbb{R}^m \to \mathbb{R}^p.$$

   See Figure 4.3.

   Assume that A_1 is the $m \times n$ standard matrix for T_1 and that A_2 is the $p \times m$ standard matrix for T_2. Consider

   $$T_1\colon \mathbf{x} \mapsto \mathbf{y} = T_1(\mathbf{x}) = A_1\mathbf{x} \in \mathbb{R}^m \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \quad \text{and}$$
   $$T_2\colon \mathbf{y} \mapsto \mathbf{z} = T_2(\mathbf{y}) = A_2\mathbf{y} \in \mathbb{R}^p,$$

   where $\mathbf{y}$ is the image of $\mathbf{x}$ under T_1 and $\mathbf{z}$ is the image of $\mathbf{y}$ under T_2. Then $\mathbf{z}$ is the image of $\mathbf{x}$ under the new linear transformation T, which is the composition of the two

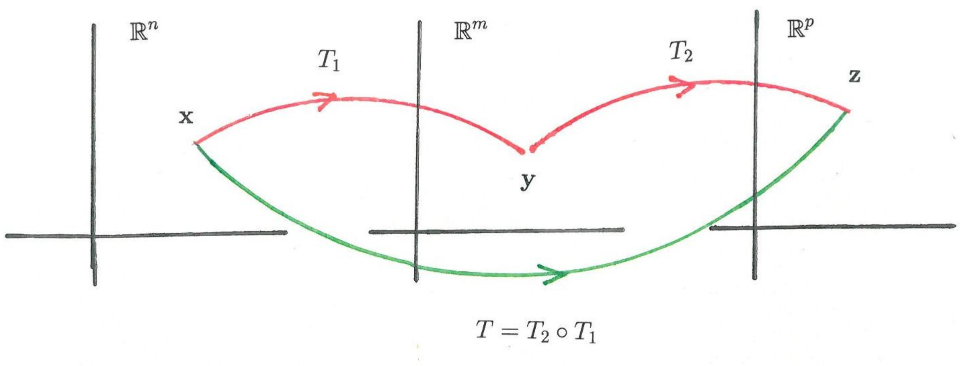

Figure 4.3: The linear composite transformation $T_2 \circ T_1$.

linear transformations T_1 followed by T_2, known as the **composite transformation**, denoted by $T_2 \circ T_1$. We write

$$T = T_2 \circ T_1 \colon \mathbb{R}^n \to \mathbb{R}^p,$$

so that, for every $\mathbf{x} \in \mathbb{R}^n$, we have

$$T = T_2 \circ T_1 \colon \mathbf{x} \mapsto T_2(T_1(\mathbf{x})) = T_2(A_1\mathbf{x}) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x} \in \mathbb{R}^p.$$

The standard matrix of the composite transformation $T_2 \circ T_1$ is the matrix product A_2A_1, which is a $p \times n$ matrix.

Problem 4.2.1.

Consider the following two linear transformations that map vectors in $\mathbb{R}^2$:

The transformation $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$, where T_1 reflects every vector in $\mathbb{R}^2$ about the line $y = 4x$.

The transformation $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$, where T_2 rotates every vector in $\mathbb{R}^2$ counter-clockwise with angle $\pi/3$ about the origin $(0, 0)$.

a) Find the standard matrix for T_1.

b) Find the standard matrix for T_2.

c) Find the standard matrix of the following composite transformations:
$T_2 \circ T_1, \quad T_1 \circ T_2, \quad T_1 \circ T_1, \quad T_2 \circ T_2.$

Solution 4.2.1.

a) **Suggestion:** Review again the Problems in **Chapter 1**, where a vector is reflected about a line.

Let A_1 denote the standard matrix for T_1, so that

$$T_1 : \mathbf{x} \mapsto A_1\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2,$$

where

$$A_1 = [T_1(\mathbf{e_1})\ T_1(\mathbf{e_2})], \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

First we find the reflection of $\mathbf{e_1}$ about the line $y = 4x$, i.e. we need to calculate $T_1(\mathbf{e_1})$:

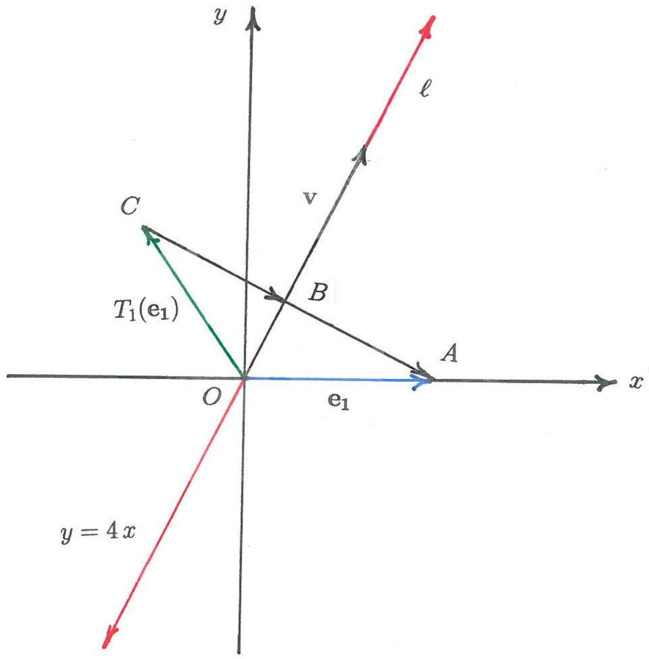

Figure 4.4: Reflection of $\mathbf{e_1}$ about $y = 4x$.

Following Figure 4.4 we have

$$T_1(\mathbf{e_1}) + \overrightarrow{CB} + \overrightarrow{BA} = \mathbf{e_1}$$

Since $\overrightarrow{CB} = \overrightarrow{BA}$, we have

$$T_1(\mathbf{e_1}) = \mathbf{e_1} - 2\overrightarrow{BA}$$

Moreover,

$$\overrightarrow{BA} = \mathbf{e_1} - \overrightarrow{OB},$$

where $\overrightarrow{OB}$ is the orthogonal projection of $\mathbf{e_1}$ onto the line ℓ given by the equation $y = 4x$, i.e. the orthogonal projection of $\mathbf{e_1}$ onto any vector on the line ℓ. To find a vector on ℓ (say $\mathbf{v}$), we let $x = 1$. Then $y = 4$, so that $\mathbf{v} = (1, 4)$ and

$$\begin{aligned}\overrightarrow{OB} &= \operatorname{proj}_{\mathbf{v}} \mathbf{e_1} = \frac{\mathbf{e_1} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\,\mathbf{v} \\ &= \frac{(1)(1) + (0)(4)}{1^2 + 4^2}\,(1, 4) \\ &= \frac{1}{17}(1, 4).\end{aligned}$$

Thus we have

$$\overrightarrow{BA} = (1, 0) - \frac{1}{17}(1, 4) = \frac{1}{17}(16, -4) \quad \text{and} \quad T_1(\mathbf{e_1}) = (1, 0) - \frac{2}{17}(16, -4) = \frac{1}{17}(-15, 8),$$

or, in column matrix form

$$T_1(\mathbf{e_1}) = \frac{1}{17}\begin{pmatrix} -15 \\ 8 \end{pmatrix}.$$

Next we find the reflection of $\mathbf{e_2}$ about the line $y = 4x$, i.e. we need to calculate $T_1(\mathbf{e_2})$:

Following Figure 4.5 we have

$$T_1(\mathbf{e_2}) = \mathbf{e_2} + \overrightarrow{AB} + \overrightarrow{BC},$$

where $\overrightarrow{AB} = \overrightarrow{BC}$. Thus

$$T_1(\mathbf{e_2}) = \mathbf{e_2} + 2\overrightarrow{AB}.$$

Moreover,

$$\overrightarrow{AB} = \overrightarrow{OB} - \mathbf{e_2},$$

where

$$\overrightarrow{OB} = \operatorname{proj}_{\mathbf{v}} \mathbf{e_2}, \quad \text{with } \mathbf{v} = (1, 4).$$

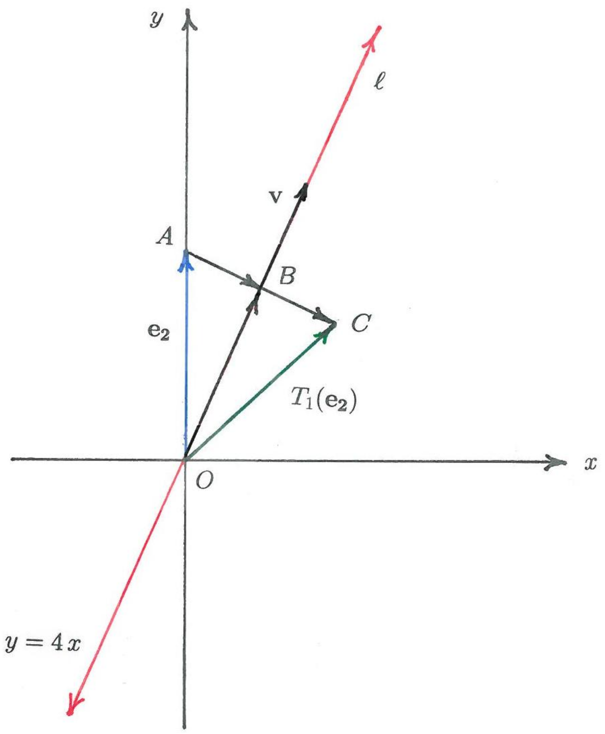

Figure 4.5: Reflection of $\mathbf{e_2}$ about $y = 4x$.

Thus

$$\overrightarrow{OB} = \operatorname{proj}_{\mathbf{v}} \mathbf{e_2} = \frac{\mathbf{e_2} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$$

$$= \frac{(0)(1) + (1)(4)}{1^2 + 4^2} (1, 4)$$

$$= \frac{4}{17}(1, 4)$$

and

$$\overrightarrow{AB} = \frac{4}{17}(1, 4) - (0, 1) = \frac{1}{17}(4, -1),$$

so that

$$T_1(\mathbf{e_2}) = (0, 1) + \frac{2}{17}(4, -1) = \frac{1}{17}(8, 15).$$

In column matrix form, we have

$$T_1(\mathbf{e_2}) = \frac{1}{17}\begin{pmatrix} 8 \\ 15 \end{pmatrix}.$$

The standard matrix A_1 for T_1 is thus

$$A_1 = \frac{1}{17}\begin{pmatrix} -15 & 8 \\ 8 & 15 \end{pmatrix}.$$

b) Let A_2 denote the standard matrix for the transformation $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$, where T_2 rotates every vector in $\mathbb{R}^2$ counter-clockwise with angle $\varphi = \pi/3$ about the origin $(0,0)$. Then

$$T_2\colon \mathbf{x} \mapsto A_2\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2,$$

where

$$A_2 = [T_2(\mathbf{e_1})\ T_2(\mathbf{e_2})], \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In Figure 4.6 we depict the counter-clockwise rotation of $\mathbf{e_1}$ and $\mathbf{e_2}$ about $(0,0)$.

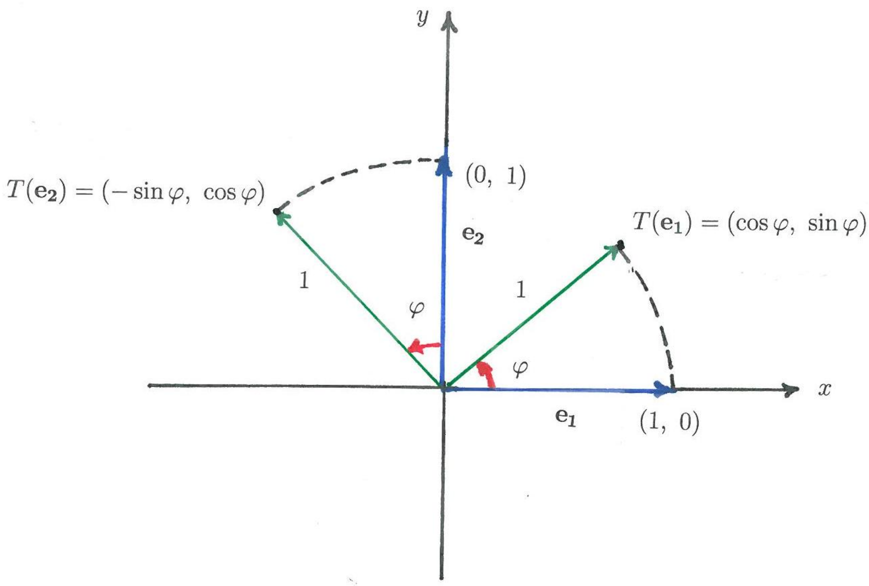

Figure 4.6: Counter-clockwise rotation with angle φ of $\mathbf{e_1}$ and $\mathbf{e_2}$ about $(0,0)$.

Following Figure 4.6 we have

$$T_2(\mathbf{e_1}) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad T_2(\mathbf{e_2}) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}.$$

Thus the standard matrix for T_2 for the counter-clockwise rotation with angle φ about $(0,0)$ is

$$A_2 = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

For the angle $\varphi = \pi/3$ we have

$$A_2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

c) The standard matrices for the listed composite transformations are given below:

$$T_2 \circ T_1 \colon \mathbf{x} \mapsto (A_2 A_1)\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2$$

$$T_1 \circ T_2 \colon \mathbf{x} \mapsto (A_1 A_2)\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2$$

$$T_1 \circ T_1 \colon \mathbf{x} \mapsto (A_1^2)\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2$$

$$T_2 \circ T_2 \colon \mathbf{x} \mapsto (A_2^2)\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

Problem 4.2.2.

Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, where T projects every vector in $\mathbb{R}^2$ orthogonally onto the line $y = k\,x$ for any $k \in \mathbb{R}$.

a) Find the standard matrix of T.

b) Let $k = -1/2$, i.e. consider the line $y = -x/2$, and find the image of the point $(1, 2)$ under T. That is, find $T(1, 2)$.

Solution 4.2.2.

a) **Suggestion:** Review again the Problems in **Chapter 1**, where a vector is projected onto another vector.

Let A denote the standard matrix for the orthogonal projection of every vector $\mathbf{x} \in \mathbb{R}^2$ onto the line $y = kx$ for all $k \in \mathbb{R}$. Then

$$T\colon \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x},$$

where

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})], \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To find $T(\mathbf{e_1})$ we need to project $\mathbf{e_1}$ orthogonally onto any position vector $\mathbf{v}$ that is lying on the line $y = kx$. See Figure 4.7.

Let $x = 1$. Then $y = k$, so that

$$T(\mathbf{e_1}) = \operatorname{proj}_{\mathbf{v}} \mathbf{e_1} = (\mathbf{e_1} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}} = \left(\frac{\mathbf{e_1} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v},$$

where

$$\mathbf{v} = (1, k) \equiv \begin{pmatrix} 1 \\ k \end{pmatrix}, \quad \hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Now

$$\mathbf{e_1} \cdot \mathbf{v} = (1, 0) \cdot (1, k) = 1$$
$$\|\mathbf{v}\|^2 = (1, k) \cdot (1, k) = 1 + k^2,$$

so that

$$T(\mathbf{e_1}) = \frac{1}{1 + k^2} \begin{pmatrix} 1 \\ k \end{pmatrix}.$$

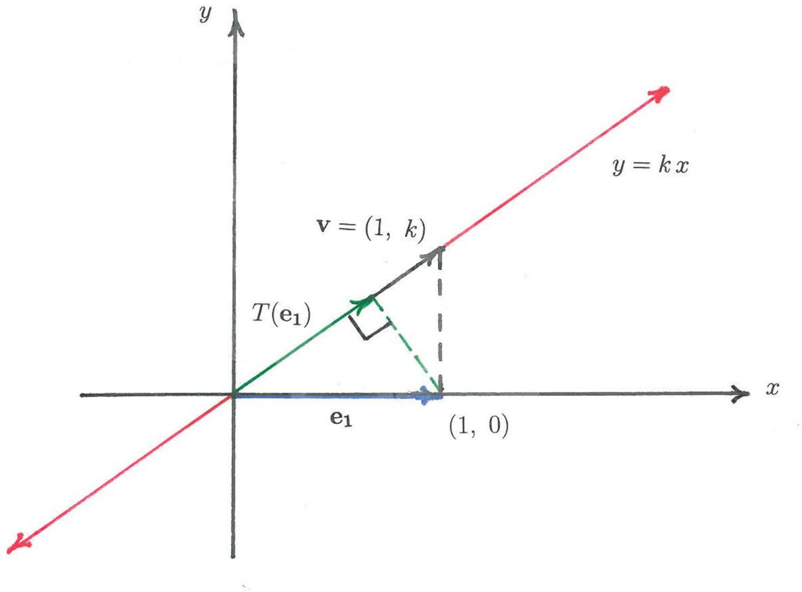

Figure 4.7: The orthogonal projection of $\mathbf{e_1}$ onto $y = kx$.

To find $T(\mathbf{e_2})$ we project $\mathbf{e_2}$ orthogonally onto vector $\mathbf{v} = (1, k)$. That is

$$T(\mathbf{e_2}) = \operatorname{proj}_{\mathbf{v}} \mathbf{e_2} = (\mathbf{e_2} \cdot \hat{\mathbf{v}})\, \hat{\mathbf{v}} = \left(\frac{\mathbf{e_2} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v},$$

where $\mathbf{e_2} \cdot \mathbf{v} = (0, 1) \cdot (1, k) = k$, so that

$$T(\mathbf{e_2}) = \frac{k}{1 + k^2} \begin{pmatrix} 1 \\ k \end{pmatrix}.$$

The standard matrix of T is therefore

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})] = \frac{1}{1 + k^2} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

b) Using the result in part a), the standard matrix for the orthogonal projection of every vector $\mathbf{x} \in \mathbb{R}^2$ onto the line $y = -x/2$ is

$$A = \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix}.$$

Then

$$T(1, 2)\colon \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mapsto A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Problem 4.2.3.

Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, where T reflects every vector in $\mathbb{R}^3$ about the line ℓ given by

$$\ell : \begin{cases} x = 2t \\ y = t \\ z = -t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

a) Find the standard matrix of T.

b) Find the image of the point $(1, 2, 3)$ under T. That is, find $T(1, 2, 3)$.

Solution 4.2.3.

a) Let A denote the standard matrix of the transformation T that reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about the line

$$\ell : \begin{cases} x = 2t \\ y = t \\ z = -t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Then

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})]$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ is the standard basis for $\mathbb{R}^3$.

To calculate $T(\mathbf{e_1})$, we project $\mathbf{e_1}$ onto any non-zero vector $\mathbf{v}$ with coordinates on the line ℓ. To find such a vector, we let $t = 1$ in the above parametric equation for ℓ and obtain

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Following Figure 4.8 we have

$$T(\mathbf{e_1}) = \mathbf{e_1} + 2\overrightarrow{AB},$$

where

$$\overrightarrow{AB} = \overrightarrow{OB} - \mathbf{e_1}$$
and
$$\overrightarrow{OB} = \operatorname{proj}_{\mathbf{v}} \mathbf{e_1} = \left(\frac{\mathbf{e_1} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

Thus we have

$$\begin{aligned} T(\mathbf{e_1}) &= 2 \operatorname{proj}_{\mathbf{v}} \mathbf{e_1} - \mathbf{e_1} \\ &= 2 \left(\frac{\mathbf{e_1} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} - \mathbf{e_1} \\ &= \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \end{aligned}$$

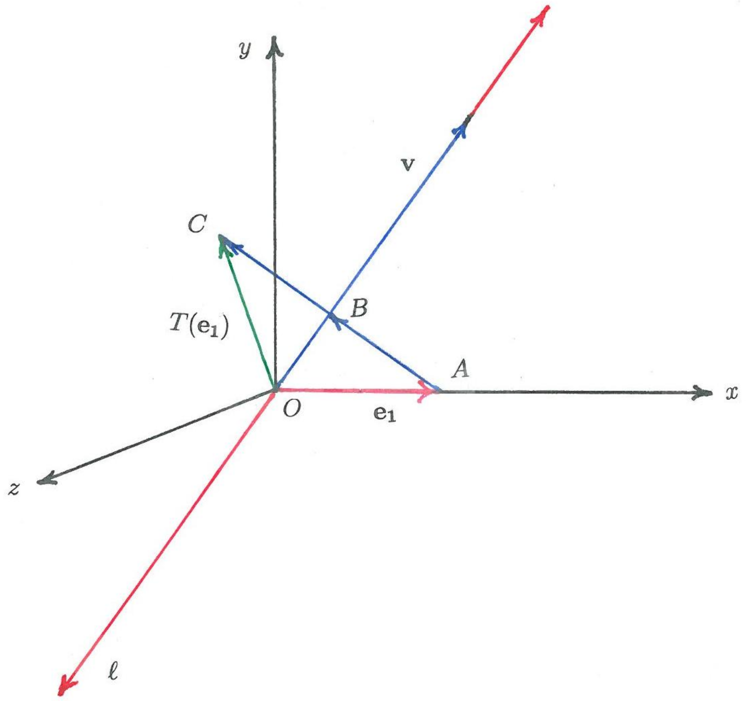

Figure 4.8: The reflection of $\mathbf{e_1}$ about the given line ℓ in $\mathbb{R}^3$.

In a similar way we find $T(\mathbf{e_2})$ and $T(\mathbf{e_3})$. We obtain

$$T(\mathbf{e_2}) = 2\,\mathrm{proj}_{\mathbf{v}}\,\mathbf{e_2} - \mathbf{e_2} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$$T(\mathbf{e_3}) = 2\,\mathrm{proj}_{\mathbf{v}}\,\mathbf{e_3} - \mathbf{e_3} = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}.$$

The standard matrix A of T is therefore

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ -2 & -1 & -2 \end{pmatrix}.$$

A point with coordinates (x, y, z) will therefore map as follows under this reflection transformation:

$$T(x,y,z)\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ -2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x+2y-2z \\ 2x-2y-z \\ -2x-y-2z \end{pmatrix}.$$

b) From part a) above we have

$$T(1,2,3)\colon \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \mapsto A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ -2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}.$$

Problem 4.2.4.

Let ℓ be a line in $\mathbb{R}^3$ that passes through the origin $(0,0,0)$. Consider now the transformation $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ that projects every vector $\mathbf{x} \in \mathbb{R}^3$ orthogonally onto ℓ as well as the transformation $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ that reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about the same line ℓ. Find the relation between the standard matrix of T_1 and the standard matrix of T_2.

Solution 4.2.4.

Let A_1 denote the standard matrix for the orthogonal projection transformation T_1 onto ℓ, i.e.

$$T_1\colon \mathbf{x} \mapsto T_1(\mathbf{x}) = A_1\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^3$$

and let A_2 denote the standard matrix for the reflection transformation T_2 about ℓ, i.e.

$$T_2\colon \mathbf{x} \mapsto T_2(\mathbf{x}) = A_2\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

As usual we consider the transformation of the standard basis vectors $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$. Referring to Figure 4.9, we have by vector addition,

$$\mathbf{e_1} + \overrightarrow{P_1Q_1} = T_1(\mathbf{e_1}) \quad \text{and} \quad \mathbf{e_1} + 2\overrightarrow{P_1Q_1} = T_2(\mathbf{e_1}).$$

Thus we obtain the relation

$$T_2(\mathbf{e_1}) = 2\,T_1(\mathbf{e_1}) - \mathbf{e_1}.$$

Referring to Figure 4.10, we have

$$\mathbf{e_2} + \overrightarrow{P_2Q_2} = T_1(\mathbf{e_2}) \quad \text{and} \quad \mathbf{e_2} + 2\overrightarrow{P_2Q_2} = T_2(\mathbf{e_2}),$$

which gives the relation

$$T_2(\mathbf{e_2}) = 2\,T_1(\mathbf{e_2}) - \mathbf{e_2}.$$

Referring to Figure 4.11, we have

$$\mathbf{e_3} + \overrightarrow{P_3Q_3} = T_1(\mathbf{e_3}) \quad \text{and} \quad \mathbf{e_3} + 2\overrightarrow{P_3Q_3} = T_2(\mathbf{e_3}),$$

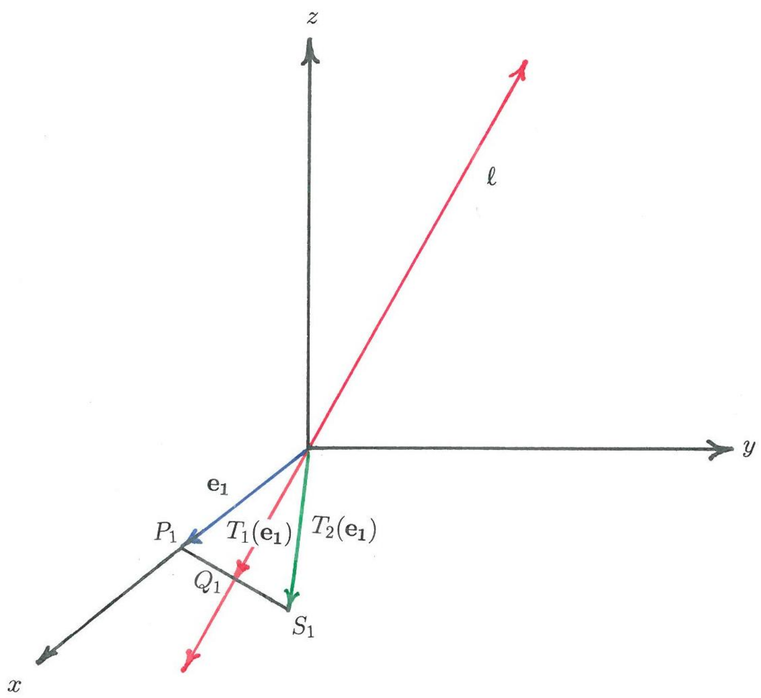

Figure 4.9: The reflection and orthogonal projection of $\mathbf{e_1}$ about the line ℓ.

which gives the relation

$$T_2(\mathbf{e_3}) = 2\,T_1(\mathbf{e_3}) - \mathbf{e_3}.$$

The standard matrix A_1 for T_1 is

$$A_1 = [T_1(\mathbf{e_1})\ T_1(\mathbf{e_2})\ T_1(\mathbf{e_3})]$$

and the standard matrix A_2 for T_2 is

$$\begin{aligned} A_2 &= [T_2(\mathbf{e_1})\ T_2(\mathbf{e_2})\ T_2(\mathbf{e_3})] \\ &= [2\,T_1(\mathbf{e_1}) - \mathbf{e_1}\ \ 2\,T_1(\mathbf{e_2}) - \mathbf{e_2}\ \ 2\,T_1(\mathbf{e_3}) - \mathbf{e_3}] \\ &= 2\,[T_1(\mathbf{e_1})\ T_1(\mathbf{e_2})\ T_1(\mathbf{e_3})] - [\mathbf{e_1}\ \mathbf{e_2}\ \mathbf{e_3}]. \end{aligned}$$

Thus the relation between A_1 and A_2 is

$$A_2 = 2\,A_1 - I_3,$$

where I_3 is the 3×3 identity matrix.

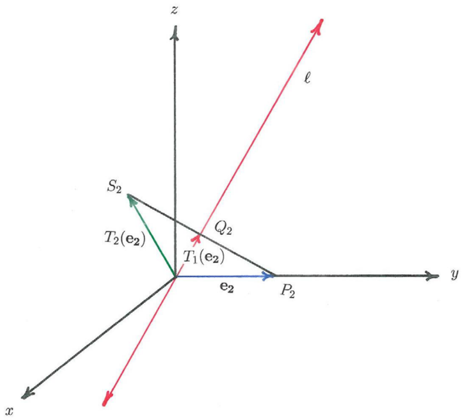

Figure 4.10: The reflection and orthogonal projection of $\mathbf{e_2}$ about the line ℓ.

Problem 4.2.5.

Let

$$A = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -5/6 & 1/(3\sqrt{2}) \\ 1/\sqrt{2} & 1/(3\sqrt{2}) & -2/3 \end{pmatrix}$$

be the standard matrix for the transformation T that reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about the line ℓ, where ℓ is a line in $\mathbb{R}^3$ that passes through the origin $(0, 0, 0)$. Find a parametric equation for ℓ.

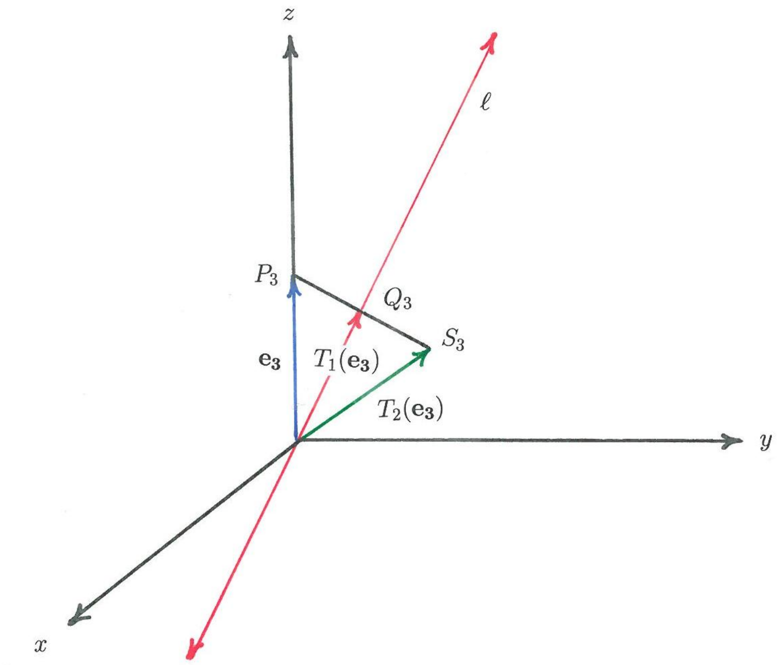

Figure 4.11: The reflection and orthogonal projection of $\mathbf{e_3}$ about the line ℓ.

Solution 4.2.5.

Since the line ℓ passes through $(0, 0, 0)$, it has the form

$$\ell : \begin{cases} x = a\,t \\ y = b\,t \\ z = c\,t \quad \text{for all } t \in \mathbb{R}, \end{cases}$$

where $\mathbf{v} = (a, b, c)$ is the direction of ℓ and this is also a vector that is lying on ℓ. We now have to find $a,\ b$ and c explicitly, such that T reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about ℓ with the given standard matrix A. For the standard basis $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$, we have

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})],$$

so that

$$T(\mathbf{e_1}) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}, \quad T(\mathbf{e_2}) = \begin{pmatrix} 1/2 \\ -5/6 \\ 1/(3\sqrt{2}) \end{pmatrix}, \quad T(\mathbf{e_3}) = \begin{pmatrix} 1/\sqrt{2} \\ 1/(3\sqrt{2}) \\ -2/3 \end{pmatrix}.$$

Referring to Figure 4.12 we have

$$\mathbf{w_1} = \text{proj}_{\mathbf{v}}\,\mathbf{e_1} \quad \text{and} \quad \mathbf{w_1} = \text{proj}_{\mathbf{v}}\,T(\mathbf{e_1}).$$

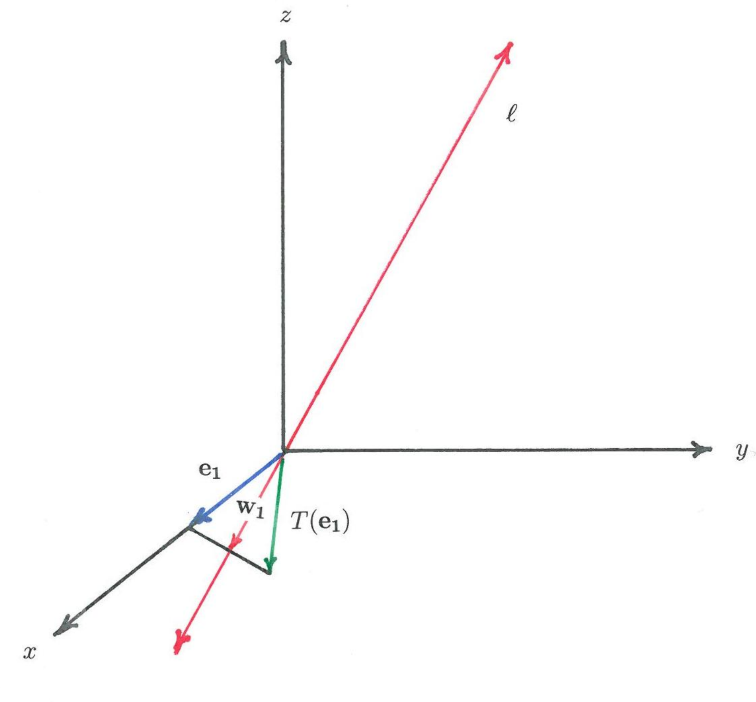

Figure 4.12: The reflection of $\mathbf{e_1}$ about the line ℓ.

This means that

$$\left(\frac{T(\mathbf{e_1})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{e_1}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \quad \text{or} \quad T(\mathbf{e_1})\cdot\mathbf{v} = \mathbf{e_1}\cdot\mathbf{v}.$$

For the above given $T(\mathbf{e_1})$ and $\mathbf{v} = (a, b, c)$ we obtain

$$\frac{1}{2}a + \frac{1}{2}b + \frac{1}{\sqrt{2}}c = a \quad \text{or} \quad a - b - \sqrt{2}c = 0.$$

Referring to Figure 4.13 we have

$$\mathbf{w_2} = \text{proj}_{\mathbf{v}}\,\mathbf{e_2} \quad \text{and} \quad \mathbf{w_2} = \text{proj}_{\mathbf{v}}\,T(\mathbf{e_2}).$$

This means that

$$\left(\frac{T(\mathbf{e_2})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{e_2}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \quad \text{or} \quad T(\mathbf{e_2})\cdot\mathbf{v} = \mathbf{e_2}\cdot\mathbf{v}.$$

For the above given $T(\mathbf{e_2})$ and $\mathbf{v} = (a, b, c)$ we obtain

$$\frac{1}{2}a - \frac{5}{6}b + \frac{1}{3\sqrt{2}}c = b \quad \text{or} \quad a - \frac{11}{3}b + \frac{\sqrt{2}}{3}c = 0.$$

Referring to Figure 4.14 we have

$$\mathbf{w_3} = \text{proj}_{\mathbf{v}}\,\mathbf{e_3} \quad \text{and} \quad \mathbf{w_3} = \text{proj}_{\mathbf{v}}\,T(\mathbf{e_3}).$$

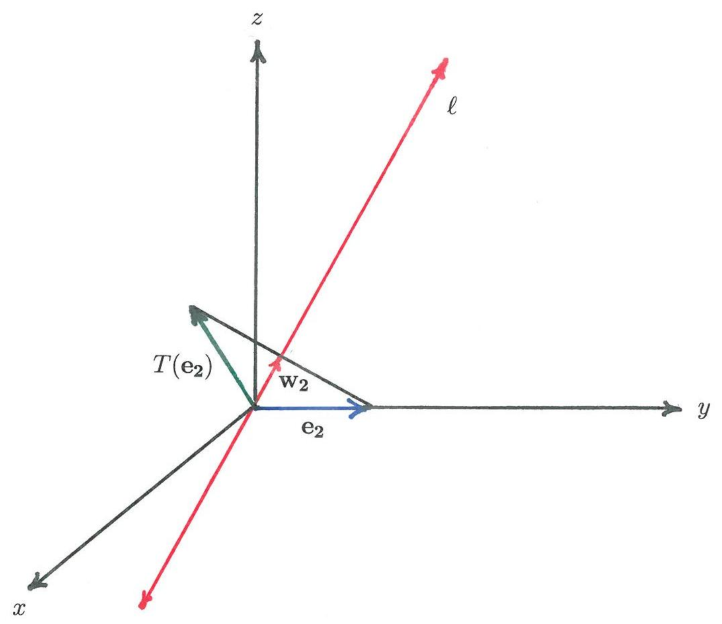

Figure 4.13: The reflection of $\mathbf{e_2}$ about the line ℓ.

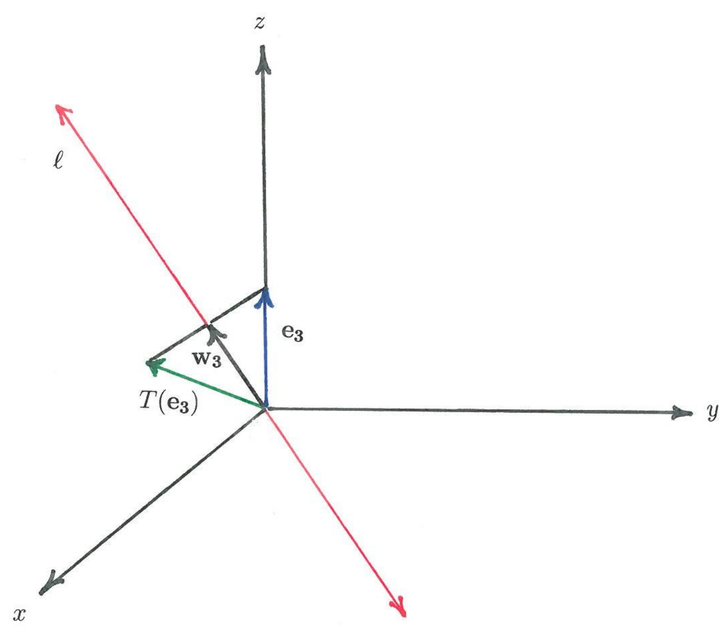

Figure 4.14: The reflection of $\mathbf{e_3}$ about the line ℓ.

This means that

$$\left(\frac{T(\mathbf{e_3})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{e_3}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \quad \text{or} \quad T(\mathbf{e_3})\cdot\mathbf{v} = \mathbf{e_3}\cdot\mathbf{v}.$$

For the above given $T(\mathbf{e_3})$ and $\mathbf{v} = (a, b, c)$ we obtain

$$\frac{1}{\sqrt{2}}a + \frac{1}{3\sqrt{2}}b - \frac{2}{3}c = c \quad \text{or} \quad a + \frac{1}{3}b - \frac{5\sqrt{2}}{3}c = 0.$$

Thus we now have three conditions for the unknown constants a, b anb c, namely

$$\begin{aligned} &a - b - \sqrt{2}c = 0 \\ &a - \frac{11}{3}b + \frac{\sqrt{2}}{3}c = 0 \\ &a + \frac{1}{3}b - \frac{5\sqrt{2}}{3}c = 0, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} 1 & -1 & -\sqrt{2} \\ 1 & -11/3 & \sqrt{2}/3 \\ 1 & 1/3 & -5\sqrt{2}/3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system by Gauss elimination we obtain the solution

$$a = \frac{3}{\sqrt{2}}\,s, \quad b = \frac{1}{\sqrt{2}}\,s, \quad c = s,$$

where s is a free parameter. We let $s = \sqrt{2}$, so the parametric equation of ℓ becomes

$$\ell : \begin{cases} x = 3t \\ y = t \\ z = \sqrt{2}\,t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Problem 4.2.6.

Find the standard matrix for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, where T projects every vector in $\mathbb{R}^3$ orthogonally onto the xy-plane.

Solution 4.2.6.

The standard matrix A for the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that projects every $\mathbf{x} \in \mathbb{R}^3$ orthogonally onto the xy-plane is

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})]$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ is the standard basis for $\mathbb{R}^3$. From Figure 4.15, it should be clear that

$$T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T(\mathbf{e_2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(\mathbf{e_3}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

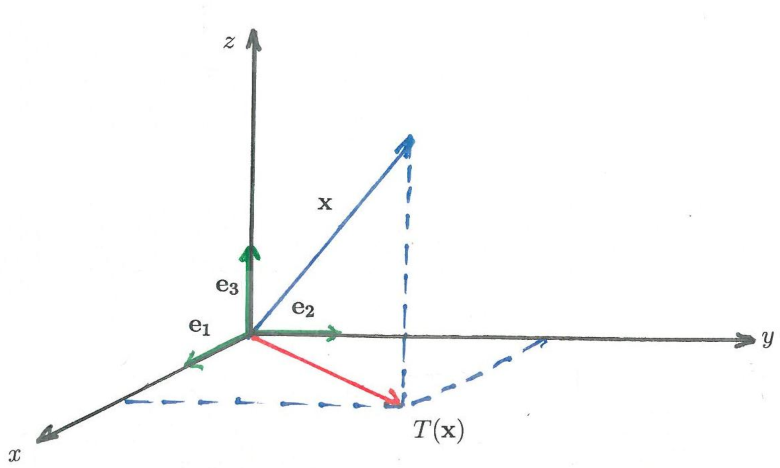

Figure 4.15: The orthogonal projection of $\mathbf{x} \in \mathbb{R}^3$ onto the xy-plane.

Thus the standard matrix A is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 4.2.7.

Find the standard matrix for $T\colon \mathbb{R}^n \to \mathbb{R}^n$, such that

$$T\colon \mathbf{x} \mapsto k\,\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^n$ and any $k \in \mathbb{R}$.

Solution 4.2.7.

We seek the $n \times n$ matrix A, such that

$$T\colon \mathbf{x} \mapsto A\mathbf{x} = k\,\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^n$. Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then

$$\begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} k\,x_1 \\ k\,x_2 \\ \vdots \\ k\,x_n \end{pmatrix} \quad \text{or} \quad kI_n\mathbf{x} = k\,\mathbf{x}.$$

Hence the standard matrix of T is $A = k\,I_n$, where I_n is the $n \times n$ identity matrix.

Problem 4.2.8.

Find the standard matrix for $T\colon \mathbb{R}^2 \to \mathbb{R}^4$, such that

$$T\colon \mathbf{x} \mapsto (k_1x_1,\ k_2x_2,\ (k_1 - k_2)x_1,\ (k_1 + k_2)x_2)$$

for every $\mathbf{x} = (x_1,\ x_2) \in \mathbb{R}^2$ and any real numbers k_1 and k_2.

Solution 4.2.8.

We have the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^4$, such that every $\mathbf{x} = (x_1,\ x_2) \in \mathbb{R}^2$ is mapped to the vector $(k_1x_1,\ k_2x_2,\ (k_1 - k_2)x_1,\ (k_1 + k_2)x_2) \in \mathbb{R}^4$ for any $k_1,\ k_2 \in \mathbb{R}$. Thus we seek the 4×2 matrix A, such that

$$T\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1\,x_1 \\ k_2\,x_2 \\ (k_1 - k_2)\,x_1 \\ (k_1 + k_2)\,x_2 \end{pmatrix}.$$

Since

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \\ k_1 - k_2 & 0 \\ 0 & k_1 + k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1\,x_1 \\ k_2\,x_2 \\ (k_1 - k_2)\,x_1 \\ (k_1 + k_2)\,x_2 \end{pmatrix},$$

it is clear that the standard matrix of T is

$$A = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \\ k_1 - k_2 & 0 \\ 0 & k_1 + k_2 \end{pmatrix}.$$

Problem 4.2.9.

Consider three linear transformations, T_1, T_2 and T_3, which map all vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$ as follows:
T_1 rotates every vector in $\mathbb{R}^3$ counter-clockwise by angle θ_1 about the z-axis;
T_2 rotates every vector in $\mathbb{R}^3$ counter-clockwise by angle θ_2 about the y-axis;
T_3 rotates every vector in $\mathbb{R}^3$ counter-clockwise by angle θ_3 about the x-axis.

a) Find the standard matrices for T_1, T_2 and T_3.

b) Find the standard matrix for the composite transformation $T = T_3 \circ T_2 \circ T_1$.

c) Consider a vector $\mathbf{u} = (x, y, z)$, where (x, y, z) is a point on the sphere with centre at $(0, 0, 0)$ and radius $a > 0$. Calculate $T(\mathbf{u})$, where T is the composite transformation in part b) and show that $T(\mathbf{u})$ is a vector with coordinates on the same sphere.

Solution 4.2.9.

a) Let $T_1 : \mathbf{x} \mapsto A_1\mathbf{x}$ denote the transformation that rotates every vector $\mathbf{x} \in \mathbb{R}^3$ counter-clockwise about the z-axis by the angle θ_1. Then

$$A_1 = [T_1(\mathbf{e_1})\ T_1(\mathbf{e_2})\ T_1(\mathbf{e_3})],$$

where

$$T_1(\mathbf{e_1}) = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix}, \quad T_1(\mathbf{e_2}) = \begin{pmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{pmatrix}, \quad T_1(\mathbf{e_3}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus the standard matrix for T_1 is

$$A_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $T_2 : \mathbf{x} \mapsto A_2\mathbf{x}$ denote the transformation that rotates every vector $\mathbf{x} \in \mathbb{R}^3$ counter-clockwise about the y-axis by the angle θ_2. Then

$$A_2 = [T_2(\mathbf{e_1})\ T_2(\mathbf{e_2})\ T_2(\mathbf{e_3})],$$

where

$$T_2(\mathbf{e_1}) = \begin{pmatrix} \cos\theta_2 \\ 0 \\ \sin\theta_2 \end{pmatrix}, \quad T_2(\mathbf{e_2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T_2(\mathbf{e_3}) = \begin{pmatrix} -\sin\theta_2 \\ 0 \\ \cos\theta_2 \end{pmatrix}.$$

Thus the standard matrix for T_2 is

$$A_2 = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$

Let $T_3 \colon \mathbf{x} \mapsto A_3\mathbf{x}$ denote the transformation that rotates every vector $\mathbf{x} \in \mathbb{R}^3$ counter-clockwise about the x-axis by the angle θ_3. Then

$$A_3 = [T_3(\mathbf{e_1})\ T_3(\mathbf{e_2})\ T_3(\mathbf{e_3})],$$

where

$$T_3(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T_3(\mathbf{e_2}) = \begin{pmatrix} 0 \\ \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}, \quad T_3(\mathbf{e_3}) = \begin{pmatrix} 0 \\ -\sin\theta_3 \\ \cos\theta_3 \end{pmatrix}.$$

Thus the standard matrix for T_3 is

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix}.$$

b) For the composite transformation

$$T = T_3 \circ T_2 \circ T_1, \qquad T \colon \mathbf{x} \mapsto A\mathbf{x}$$

the standard matrix A is

$$A = A_3\, A_2\, A_1,$$

where $A_1,\ A_2$ and A_3 are given in part a) above. Thus

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_1\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3 & -\cos\theta_2\sin\theta_3 \\ \sin\theta_1\sin\theta_3 + \cos\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_2\cos\theta_3 \end{pmatrix}.$$

c) Let $\mathbf{u}$ be a vector on the sphere with radius $a > 0$, given by the equation

$$x^2 + y^2 + z^2 = a^2.$$

Then $\mathbf{u}$ has the following coordinates:

$$\mathbf{u} = (x,\ y,\ \sqrt{a^2 - x^2 - y^2}).$$

We now map $\mathbf{u}$ by $T = T_3 \circ T_2 \circ T_1$, i.e.

$$T\colon \mathbf{u} \mapsto T(\mathbf{u}) = A\mathbf{u},$$

where A is the standard matrix given in part b). This leads to

$$A\mathbf{u} = \mathbf{w} = (w_1,\ w_2,\ w_3),$$

where

$$\begin{aligned}
w_1 &= x\cos\theta_1\cos\theta_2 - y\sin\theta_1\cos\theta_2 - \sqrt{a^2 - x^2 - y^2}\,\sin\theta_2 \\
w_2 &= x\,(\sin\theta_1\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3) + y\,(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3) \\
&\quad - \sqrt{a^2 - x^2 - y^2}\,\cos\theta_2\sin\theta_3 \\
w_3 &= x\,(\sin\theta_1\sin\theta_3 + \cos\theta_1\sin\theta_2\cos\theta_3) + y\,(\cos\theta_1\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3) \\
&\quad + \sqrt{a^2 - x^2 - y^2}\,\cos\theta_2\cos\theta_3.
\end{aligned}$$

We calculate $w_1^2 + w_2^2 + w_3^2$ and obtain

$$w_1^2 + w_2^2 + w_3^2 = a^2,$$

which shows that $\mathbf{w}$ is a vector on the sphere with radius $a > 0$ and centre $(0, 0, 0)$.

Problem 4.2.10.

a) Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, where T projects every vector $\mathbf{x} \in \mathbb{R}^3$ orthogonally onto the plane

$$\Pi:\ ax + by + cz = 0.$$

Find the standard matrix for T.

b) Find a parametric equation of the line $\hat{\ell}$, where $\hat{\ell}$ is the orthogonal projection of the line

$$\ell : \begin{cases} x = t + 2 \\ y = -t + 1 \\ z = 3t - 1 \quad \text{for all } t \in \mathbb{R} \end{cases}$$

onto the plane

$$\Pi : \; x + 2y - 3z = 0.$$

Solution 4.2.10.

a) Let A be the standard matrix of T. For the standard basis $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ we then have

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})].$$

The normal vector $\mathbf{n}$ of the plane Π is

$$\mathbf{n} = (a, b, c).$$

Referring to Figure 4.16 we have

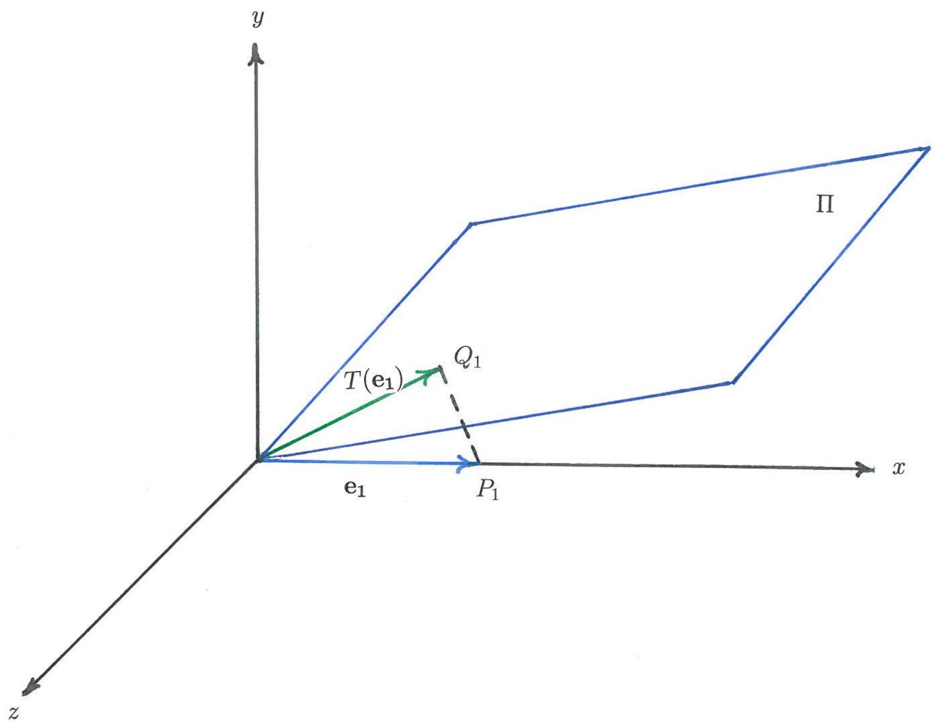

Figure 4.16: The orthogonal projection of $\mathbf{e_1}$ onto the plane Π.

$$T(\mathbf{e_1}) + \overrightarrow{Q_1P_1} = \mathbf{e_1},$$

where

$$\overrightarrow{Q_1P_1} = \mathrm{proj}_{\mathbf{n}}\,\mathbf{e_1} = \left(\frac{\mathbf{e_1}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \frac{a}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Therefore

$$T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{a}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 \\ -ab \\ -ac \end{pmatrix}.$$

Referring to Figure 4.17 we have

$$T(\mathbf{e_2}) + \overrightarrow{Q_2P_2} = \mathbf{e_2},$$

where

$$\overrightarrow{Q_2P_2} = \text{proj}_{\mathbf{n}}\,\mathbf{e_2} = \left(\frac{\mathbf{e_2}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \frac{b}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

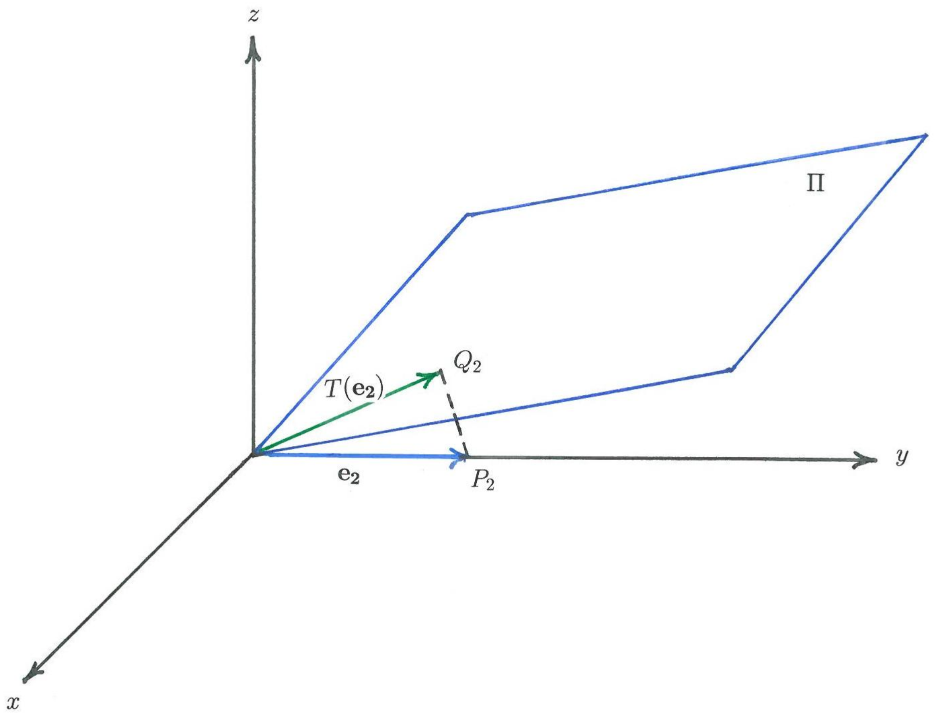

Figure 4.17: The orthogonal projection of $\mathbf{e_2}$ onto the plane Π.

Therefore

$$T(\mathbf{e_2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{b}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -ab \\ a^2+c^2 \\ -bc \end{pmatrix}.$$

Referring to Figure 4.18 we have

$$T(\mathbf{e_3}) + \overrightarrow{Q_3P_3} = \mathbf{e_3},$$

where

$$\overrightarrow{Q_3P_3} = \text{proj}_{\mathbf{n}}\, \mathbf{e_3} = \left(\frac{\mathbf{e_3}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \frac{c}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

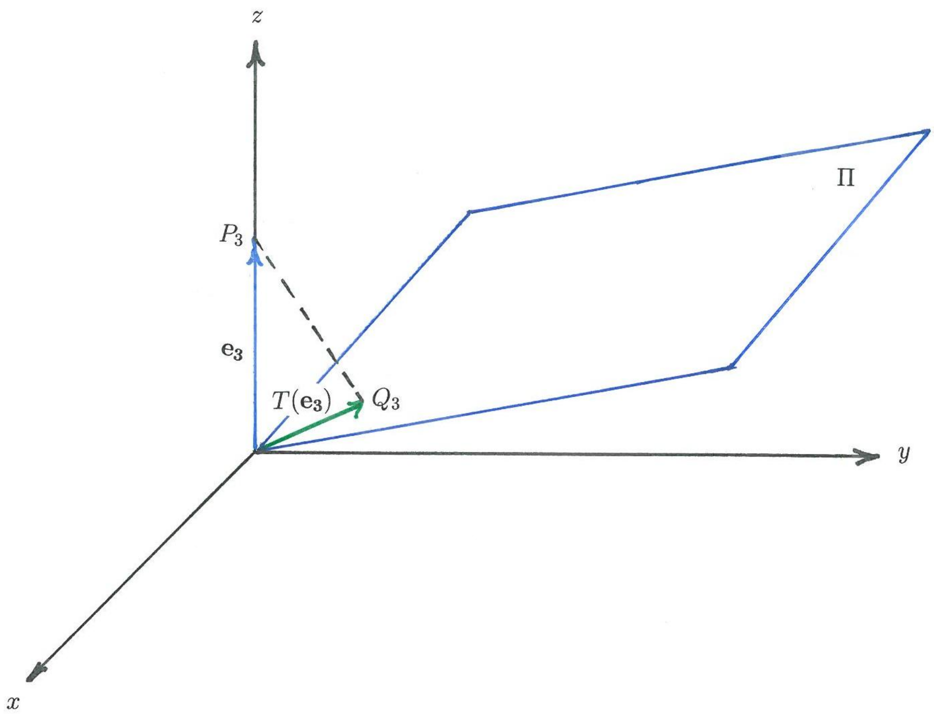

Figure 4.18: The orthogonal projection of $\mathbf{e_3}$ onto the plane Π.

Therefore

$$T(\mathbf{e_3}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{c}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}.$$

The standard matrix A for T is thus

$$A = [T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})] = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}.$$

b) To find the projection of the given line ℓ onto the given plane Π, we choose any two points P and Q on ℓ and project their position vectors onto Π using the standard matrix that was dertived in part a). We refer to Figure 4.19.

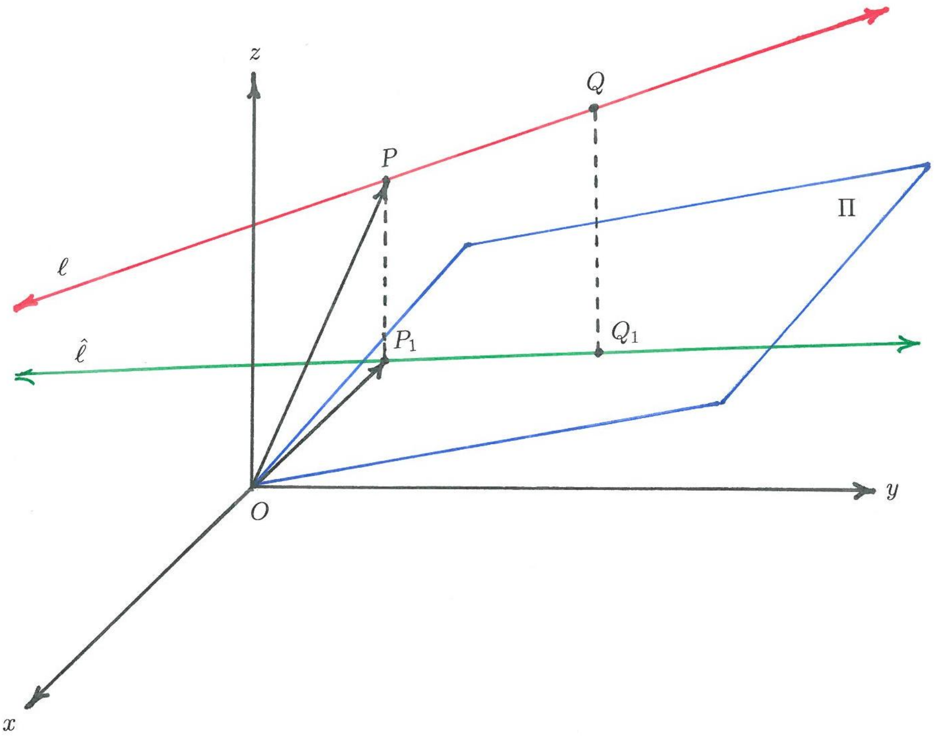

Figure 4.19: The orthogonal projection of ℓ onto the plane Π.

We choose the following two points on ℓ: $P : (2, 1, -1)$ that corresponds to the parameter value $t = 0$ and $Q : (3, 0, 2)$ that corresponds to $t = 1$. Using the standard matrix A of the orthogonal projection of any vector in $\mathbb{R}^3$ onto the plane Π given in part a), we obtain for our plane

$$\Pi : \ x + 2y - 3z = 0$$

the standard matrix

$$A = \frac{1}{14}\begin{pmatrix} 13 & -2 & 3 \\ -2 & 10 & 6 \\ 3 & 6 & 5 \end{pmatrix}.$$

Now

$$\overrightarrow{OP_1} = \frac{1}{14}\begin{pmatrix} 13 & -2 & 3 \\ -2 & 10 & 6 \\ 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\overrightarrow{OQ_1} = \frac{1}{14}\begin{pmatrix} 13 & -2 & 3 \\ -2 & 10 & 6 \\ 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 45 \\ 6 \\ 19 \end{pmatrix}.$$

We now have vector $\overrightarrow{P_1Q_1}$, namely

$$\overrightarrow{P_1Q_1} = (\frac{45}{14} - \frac{3}{2}, \frac{3}{7} - 0, \frac{19}{14} - \frac{1}{2}) = (\frac{12}{7}, \frac{3}{7}, \frac{6}{7}).$$

The vector $\overrightarrow{P_1Q_1}$ gives the direction of the line $\hat{\ell}$ and, using the point P_1, we obtain the following parametrized equation of the line $\hat{\ell}$:

$$\hat{\ell} : \begin{cases} x = \frac{12}{7}t + \frac{3}{2} \\ y = \frac{3}{7}t \\ z = \frac{6}{7}t + \frac{1}{2} \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

4.3 Invertible linear transformations

In this section we discuss surjective and injective linear transformations and study invertible linear transformations.

Theoretical Remarks 4.3.

1. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, where

$$T\colon \mathbf{x} \mapsto \mathbf{b}$$

is said to be **surjective** onto a subset W of $\mathbb{R}^m$ (or just **onto** W), if each vector $\mathbf{b} \in W$ is the image of at least one vector $\mathbf{x} \in \mathbb{R}^n$.

2. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, where

$$T\colon \mathbf{x} \mapsto \mathbf{b},$$

is said to be **injective** on a subset W of $\mathbb{R}^m$ (or just **one-to-one** on W), if each vector $\mathbf{b} \in W$ is the image of exactly one vector $\mathbf{x} \in \mathbb{R}^n$.

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A, i.e.

$$T\colon \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x} \in \mathbb{R}^m,$$

where A is an $m \times n$ matrix. Then we have the following

Properties:

a) T is injective on its range R_T if and only if

$$A\mathbf{x} = \mathbf{0}$$

has only the zero-solution $\mathbf{x} = \mathbf{0}$.

b) T is injective on its range R_T if and only if the columns of A form a linearly independent set of n vectors in $\mathbb{R}^m$. Then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution $\mathbf{x} \in \mathbb{R}^n$.

c) T is surjective onto $\mathbb{R}^m$ if and only if the co-domain of T, namely $\mathbb{R}^m$, is the range R_T of T, i.e. if and only if $R_T = \mathbb{R}^m$.

d) If T is injective on a set, then T is surjective onto this set.

Remark: *Any linear transformation* $T\colon \mathbb{R}^n \to \mathbb{R}^m$ *with standard matrix* $A = [\mathbf{a_1}\ \mathbf{a_2}\ \cdots\ \mathbf{a_n}]$ *is always surjective onto its range* $R_T \subseteq \mathbb{R}^m$ *and then*

$$R_T = \mathit{span}\{\mathbf{a_1},\ \mathbf{a_2},\ \cdots,\ \mathbf{a_n}\}.$$

If there exist vectors in the co-domain $\mathbb{R}^m$ *that are not in* R_T*, then* $R_T \neq R^m$ *and then* T *is obviously not surjective onto* $\mathbb{R}^m$*.*

4. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an injective linear transformation on $\mathbb{R}^n$ with $n \times n$ standard matrix A. Assume now that there exists another injective linear transformation $T^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$, such that

$$T^{-1} \circ T(\mathbf{x}) = T \circ T^{-1}(\mathbf{x}) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Then T^{-1} is the inverse of T and the standard matrix of T^{-1} is the inverse matrix A^{-1} of A. That is

$$T^{-1}\colon \mathbf{x} \mapsto T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Problem 4.3.1.

Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

a) Give the domain $\mathcal{D}_T$ and the range R_T of the transformation T.

b) Is the transformatin T surjective and/or injective onto its range R_T? Explain.

c) Is the transformation surjective onto $\mathbb{R}^2$? Explain.

Solution 4.3.1.

a) We are given the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

That is

$$T\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$$

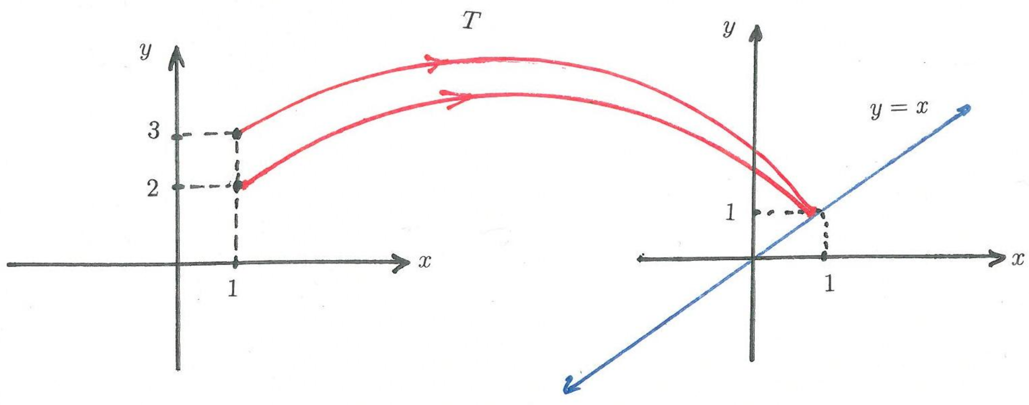

Figure 4.20: A surjective transformation T onto $y = x$ that is not injective.

for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$. See Figure 4.20.

The domain $\mathcal{D}_T$ of T therefore consists of all vectors in $\mathbb{R}^2$, i.e.

$$\mathcal{D}_T = \{\begin{pmatrix} x \\ y \end{pmatrix} \text{ for all } x,\ y \in \mathbb{R}\}$$

$$= \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = \mathbb{R}^2.$$

The range R_T of T consists of all those vectors in $\mathbb{R}^2$ which lie on the line $y = x$, i.e

$$R_T = \{\begin{pmatrix} k \\ k \end{pmatrix} \text{ for all } k \in \mathbb{R}\}$$

$$= \text{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \subset \mathbb{R}^2.$$

b) The transformation is surjective onto its range R_T (the line $y = x$), as every vector in R_T is the image of at least one vector in the domain $\mathcal{D}_T = \mathbb{R}^2$. However, T is not injective, as there exist more than one vector in $\mathbb{R}^2$ that map to the same vector in R_T. In fact there exist infinitely many vectors in $\mathbb{R}^2$ that map to the same point in R_T, for every point in R_T. For example, both the vectors $(1, 2)$ and $(1, 3)$ are mapped to the vector $(1, 1)$ by T. Moreover, the vectors $(1, k)$ are all mapped to $(1, 1)$ for all $k \in \mathbb{R}$.

c) The transformation T is not surjective onto $\mathbb{R}^2$, since only the vectors on the line $y = x$ are images under T. So not every vector in $\mathbb{R}^2$ is an image under T. For example, the vector $\mathbf{v} = (1, 2)$ is not an image under T for any point in $\mathbb{R}^2$.

Problem 4.3.2.

Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$T\colon (x_1,\ x_2,\ x_3) \mapsto (x_1 - x_2 + 5x_3,\ x_1 + 2x_2 - 4x_3,\ 2x_1 + 3x_2 - 5x_3).$$

a) Prove that T is a linear transformation.

b) Find the standard matrix of T.

c) What is the domain $\mathcal{D}_T$ and the range R_T of T. Give $\mathcal{D}_T$ and R_T in terms of spanning sets.

d) Is T surjective onto $\mathbb{R}^3$? Explain.

e) Is T injective on its range R_T? Explain.

Solution 4.3.2.

a) We prove that the transformation T, given by

$$T\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 + 5x_3 \\ x_1 + 2x_2 - 4x_3 \\ 2x_1 + 3x_2 - 5x_3 \end{pmatrix},$$

is a linear transformation. Consider two vectors in $\mathbb{R}^3$, namely $\mathbf{x} = (x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_1,\ y_2,\ y_3)$. Then

$$T(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 + 5x_3 \\ x_1 + 2x_2 - 4x_3 \\ 2x_1 + 3x_2 - 5x_3 \end{pmatrix}, \quad T(\mathbf{y}) = \begin{pmatrix} y_1 - y_2 + 5y_3 \\ y_1 + 2y_2 - 4y_3 \\ 2y_1 + 3y_2 - 5y_3 \end{pmatrix}.$$

We need to show that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and that $T(c\,\mathbf{x}) = c\,T(\mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ and all $c \in \mathbb{R}$. We have

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= \begin{pmatrix} x_1 + y_1 - (x_2 + y_2) + 5(x_3 + y_3) \\ x_1 + y_1 + 2(x_2 + y_2) - 4(x_3 + y_3) \\ 2(x_1 + y_1) + 3(x_2 + y_2) - 5(x_3 + y_3) \end{pmatrix} \\ &= \begin{pmatrix} x_1 - x_2 + 5x_3 \\ x_1 + 2x_2 - 4x_3 \\ 2x_1 + 3x_2 - 5x_3 \end{pmatrix} + \begin{pmatrix} y_1 - y_2 + 5y_3 \\ y_1 + 2y_2 - 4y_3 \\ 2y_1 + 3y_2 - 5y_3 \end{pmatrix} \\ &= T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

Furthermore, we have

$$T(c\,\mathbf{x}) = \begin{pmatrix} cx_1 - cx_2 + 5cx_3 \\ cx_1 + 2cx_2 - 4cx_3 \\ 2cx_1 + 3cx_2 - 5cx_3 \end{pmatrix} = c \begin{pmatrix} x_1 - x_2 + 5x_3 \\ x_1 + 2x_2 - 4x_3 \\ 2x_1 + 3x_2 - 5x_3 \end{pmatrix} = c\,T(\mathbf{x}) \quad \text{for all } c \in \mathbb{R}.$$

Since $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors in $\mathbb{R}^3$, the above two properties of T hold for all vectors in $\mathbb{R}^3$. This proves that T is a linear transformation.

b) The standard matrix of T is a 3×3 matrix A, such that

$$T\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + 5x_3 \\ x_1 + 2x_2 - 4x_3 \\ 2x_1 + 3x_2 - 5x_3 \end{pmatrix}.$$

Thus the standard matrix is

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 1 & 2 & -4 \\ 2 & 3 & -5 \end{pmatrix}.$$

c) The domain $\mathcal{D}_T$ of T consists of all the vectors in $\mathbb{R}^3$, since T maps every vector in $\mathbb{R}^3$. That is

$$\mathcal{D}_T = \text{span}\,\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\},$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ are the standard basis vectors of $\mathbb{R}^3$.

To establish the range R_T of T, we need to find all vectors $\mathbf{b} \in \mathbb{R}^3$, such that the system

$$A\mathbf{x} = \mathbf{b}$$

is consistent, where A is the standard matrix of T, namely

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 1 & 2 & -4 \\ 2 & 3 & -5 \end{pmatrix}.$$

Let

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Then the associated augmented matrix of the above linear system is

$$[A\ \mathbf{b}] = \begin{pmatrix} 1 & -1 & 5 & b_1 \\ 1 & 2 & -4 & b_2 \\ 2 & 3 & -5 & b_3 \end{pmatrix}.$$

Applying elementary row operations to this augmented matrix, we obtain the following row equivalent matrices:

$$\begin{pmatrix} 1 & -1 & 5 & b_1 \\ 1 & 2 & -4 & b_2 \\ 2 & 3 & -5 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 5 & b_1 \\ 0 & 3 & -9 & b_2 - b_1 \\ 0 & 5 & -15 & b_3 - 2b_1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 5 & b_1 \\ 0 & 1 & -3 & b_2/3 - b_1/3 \\ 0 & 0 & 0 & -5b_2/3 - b_1/3 + b_3 \end{pmatrix}$$

By the third row of the last row equivalent matrix, we conclude that the given linear system is consistent if and only if $-5b_2/3 - b_1/3 + b_3 = 0$, or multyplying this equation by 3, we have the following condition on the coordinates of vector $\mathbf{b}$:

$$-5b_2 - b_1 + 3b_3 = 0.$$

Thus

$$\mathbf{b} = \begin{pmatrix} -5b_2 + 3b_3 \\ b_2 \\ b_3 \end{pmatrix} = b_2 \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{for all } b_2 \in \mathbb{R} \text{ and all } b_3 \in \mathbb{R}.$$

Thus the range of T is a plane in $\mathbb{R}^3$ that passes through the origin $(0, 0, 0)$ and that is spanned as follows:

$$R_T = \text{span}\left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

d) The given transformation T is not surjective onto $\mathbb{R}^3$, as there are vectors in $\mathbb{R}^3$ that are not images under T. In fact, any vector in $\mathbb{R}^3$ that is not lying on the plane spanned as given by R_T in part c) above, is not an image under T.

e) The given transformation T is not injective on R_T, as for every vector $\mathbf{b} \in R_T$ there exist more than one (in fact infinitely many) vectors $\mathbf{x} \in \mathbb{R}^3$ that map to this image vector $\mathbf{b}$. We know this from the fact that the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions $\mathbf{x}$, with x_3 being a free parameter for every $\mathbf{b} \in R_T$, namely every vector $\mathbf{b}$ of the form

$$\mathbf{b} = \begin{pmatrix} -5b_2 + 3b_3 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{for any real } b_2 \text{ and } b_3.$$

Problem 4.3.3.

Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with the standard matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ -3 & -2 & 4 \end{pmatrix}.$$

a) Find all vectors $\mathbf{x} \in \mathbb{R}^3$, such that $T(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$.

b) Is the given transformation T invertible? If so, find the standard matrix for T^{-1}.

Solution 4.3.3.

a) Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and find $\mathbf{x}$, such that

$$T\colon \mathbf{x} \mapsto A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

for the given standard matrix A. We therefore need to solve the linear system

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ -3 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

The corresponding augmented matrix, and some of its row equivalent matrices, are

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 2 \\ -3 & -2 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & -5 & 10 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

The last row equivalent matrix given above is the reduced echelon form of A. Thus we have the unique solution of the linear system, namely

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \quad \text{for} \quad T(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

b) Since

$$\det A = 10$$

we know that the matrix A is invertible and therefore we know that T is an injective linear transformation with range $R_T = \mathbb{R}^3$. This means that T is an invertible transformation on $\mathbb{R}^3$ and that the standard matrix for its inverse T^{-1} is A^{-1}. We therefore need to calculate A^{-1}. For that, we consider $[A\ I_3]$, where I_3 is the 3×3 identity matrix. We obtain

$$[A\ I_3] \sim \begin{pmatrix} 1 & 0 & 0 & 2/5 & 0 & -1/5 \\ 0 & 1 & 0 & -7/5 & 1 & 1/5 \\ 0 & 0 & 1 & -2/5 & 1/2 & 1/5 \end{pmatrix}.$$

Thus the inverse matrix of A is

$$A^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -7/5 & 1 & 1/5 \\ -2/5 & 1/2 & 1/5 \end{pmatrix},$$

so that

$$T^{-1} \colon \mathbf{x} \mapsto A^{-1}\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

Problem 4.3.4.

Consider a linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ for which the following is valid:

$$T \colon \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 5 \\ 10 \\ 4 \end{pmatrix}, \quad T \colon \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix}, \quad T \colon \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \mapsto \begin{pmatrix} 10 \\ 15 \\ 4 \end{pmatrix}.$$

a) Find the standard matrix of T.

b) Is T an invertible transformation? Explain.

Solution 4.3.4.

a) To find the standard matrix A of T, we first introduce some notations for the vectors that are involved. Let

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\mathbf{v_1} = \begin{pmatrix} 5 \\ 10 \\ 4 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 10 \\ 15 \\ 4 \end{pmatrix}.$$

Then, as given in this exercise, we have

$$T\colon \mathbf{u_1} \mapsto A\mathbf{u_1} = \mathbf{v_1}, \quad T\colon \mathbf{u_2} \mapsto A\mathbf{u_2} = \mathbf{v_2}, \quad T\colon \mathbf{u_3} \mapsto A\mathbf{u_3} = \mathbf{v_3}.$$

Now

$$A[\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}] = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}]$$

and by denoting $U = [\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}]$ and $V = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}]$, we have the matrix equation

$$AU = V,$$

where

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & -2 \end{pmatrix}, \quad V = \begin{pmatrix} 5 & 0 & 10 \\ 10 & -5 & 15 \\ 4 & 5 & 4 \end{pmatrix}.$$

Calculating the determinant of U, we obtain

$$\det U = -5,$$

which means that the columns of matrix U form a linearly independent set and that U is an invertible matrix. Thus we can solve the matrix equation for A by multiplying the equation with U^{-1} from the right. We obtain

$$A = VU^{-1}.$$

Calculating U^{-1}, we obtain

$$U^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -2 & -1 \\ 3 & 1 & 3 \\ -2 & 1 & -2 \end{pmatrix},$$

so that the standard matrix A follows:

$$A = \frac{1}{5}\begin{pmatrix} 5 & 0 & 10 \\ 10 & -5 & 15 \\ 4 & 5 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 & -1 \\ 3 & 1 & 3 \\ -2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -5 \\ -1 & -2 & -11 \\ 23/5 & 1/5 & 3/5 \end{pmatrix}.$$

b) To find out whether T is an invertible transformation, we can inverstigate the invertibility of its standard matrix A that was calculated in part a) above. We recall that A is an invertible matrix if and only if $\det A \neq 0$. We therefore calculate $\det A$ and obtain

$$\det A = -45.$$

Hence A is invertible, which makes T an invertible transformation and the standard matrix of T^{-1} is A^{-1}.

4.4 Exercises

1. Consider the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, such that every vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is mapped to $\mathbb{R}^2$ in the following manner:

$$T\colon (x_1, x_2, x_3) \mapsto (x_1 - 5x_2 + 4x_3,\ x_2 - 6x_3).$$

   a) Show that T is a linear transformation.

   [**Answer:** We need to show that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and that $T(c\,\mathbf{x}) = c\,T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$, all $\mathbf{y} \in \mathbb{R}^3$ and all $c \in \mathbb{R}$.]

   b) Find the standard matrix of T.

   [**Answer:** The standard matrix is $A = \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix}$.]

   c) Find the range of T and establish whether T is surjective onto $\mathbb{R}^2$.

   [**Answer:** The range of T is $\mathbb{R}^2$, so that T is surjective onto $\mathbb{R}^2$.]

   d) Is T an injective transformation on its range? Explain.

   [**Answer:** T is not injective, as $A\mathbf{x} = \mathbf{b}$ has infinitely many solution $\mathbf{x} \in \mathbb{R}^3$ for any $\mathbf{b} \in \mathbb{R}^2$.]

2. Consider a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, such that T maps every $\mathbf{x} = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3$ to $(k_1\,x_1 + x_3,\ k_2\,x_2 - x_3) \in \mathbb{R}^2$ for any $k_1 \in \mathbb{R}$ and any $k_2 \in \mathbb{R}$. Find the standard matrix A of T.

[**Answer:** $A = \begin{pmatrix} k_1 & 0 & 1 \\ 0 & k_2 & -1 \end{pmatrix}$.]

3. Consider a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^2$ for any $n \geq 2$, such that

$$T\colon \mathbf{x} \mapsto (\sum_{i=1}^{n} k_i\,x_i,\ x_n)$$

for every $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n) \in \mathbb{R}^n$ and any constants $k_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

a) Find the standard matrix A of T.

[**Answer:** $A = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.]

b) Let $k_j = 1$ for $j = 1, 2, \ldots, n$. Find now the image of the point

$$(1,\ 2,\ \ldots, n) \in \mathbb{R}^n$$

under T. What is this image if $n = 100$?

[**Answer:** $\left(\frac{n(n+1)}{2},\ n\right)$. For $n = 100$, we have the image $(5050,\ 100)$.]

4. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, such that

$$T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad T(\mathbf{e_2}) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad T(\mathbf{e_3}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ are the standard basis vectors of $\mathbb{R}^3$. Find the standard matrix A of T and determine $T(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

[**Answer:** $A = \begin{pmatrix} 1 & -1 & 0 \\ 5 & -2 & 1 \end{pmatrix}, \quad T(\mathbf{x}) = \begin{pmatrix} 2 \\ 17 \end{pmatrix}$.]

5. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, where T projects every vector in $\mathbb{R}^2$ orthogonally onto the line $y = -3x$.

a) Find the standard matrix A of T.

[**Answer:** $A = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$.]

b) Find $T(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

[**Answer:** $T(\mathbf{x}) = \frac{1}{5}\begin{pmatrix} -7 \\ 21 \end{pmatrix}$.]

c) Is T invertible? Explain.

[**Answer:** T is not an invertible transformation, as $\det A = 0$. This is also clear geometrically, as there are obviously inifinitely many vectors that project orthogonally onto the same point on the line $y = -3x$, for every point on $y = -3x$.]

6. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, where T reflects every vector in $\mathbb{R}^2$ about the line $y = 3x$.

a) Find the standard matrix A of T.

[**Answer:** $A = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$.]

b) Show that T is an injective transformation on $\mathbb{R}^2$ and find the standard matrix for the inverse transformation T^{-1}.

[**Answer:** Since T describes a reflection about a line, it is geometrically clear that T is injective and invertible on $\mathbb{R}^2$. This can also be established by calculating the determinant of A. We obtain $\det A = -1$. Hence A is an invertible matrix and the standard matrix of T^{-1} is A^{-1}, which is the same as the standard matrix of T, i.e. $A^{-1} = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$.]

7. Consider two linear transformations, T_1 and T_2, where both map vectors in $\mathbb{R}^2$. In particular, T_1 rotates every vector in $\mathbb{R}^2$ counter-clockwise with angle $\pi/3$ about the origin $(0,0)$ and T_2 maps every vector $\mathbf{x} = (x_1, x_2)$ as follows:

$$T_2\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} \quad \text{for all } x_1 \in \mathbb{R} \text{ and all } x_2 \in \mathbb{R}.$$

a) Find the standard matrix A_1 for T_1 and the standard matrix A_2 for T_2. Are T_1 and T_2 invertible? Explain.

[**Answer:** $A_1 = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. T_1 and T_2 are invertible.]

b) Find the standard matrix A for the composite transformation $T = T_2 \circ T_1$. Is T invertible and, if so, find the standard matrix B for T^{-1}.

[**Answer:** $A = A_2 A_1 = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, $B = A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3}-1 \\ -\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$.]

8. Consider the linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad T(\mathbf{e_2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(\mathbf{e_3}) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

where $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ are the standard basis vectors of $\mathbb{R}^3$. Establish whether T is an invertible transformation and, if so, find the standard matrix for the inverse transformation.

[**Answer:** The standard matrix A of T is $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Since $\det A = 2$, T is invertible and the standard matrix of T^{-1} is $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.]

9. Consider the following two planes in $\mathbb{R}^3$:

$$\Pi_1 : \ x - y + 3z = 0$$
$$\Pi_2 : \ 2x + y + 3z = 0.$$

a) Find the line ℓ of intersection of the given planes Π_1 and Π_2 and express ℓ in parametric form.

[**Answer:**

$$\ell : \begin{cases} x = -2t \\ y = t \\ z = t \qquad \text{for all } t \in \mathbb{R}. \end{cases}$$]

b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation that projects every vector $\mathbf{x} \in \mathbb{R}^3$ orthogonally onto the line ℓ obtained in part a) of this problem. Find the standard matrix A of T.

[**Answer:** $A = \dfrac{1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$.]

10. Consider three linear tansformations, T_1, T_2 and T_3, all of which map vectors in $\mathbb{R}^3$. In particular, T_1 projects every vector in $\mathbb{R}^3$ orthogonally onto the line ℓ, given by the following parametric equation:

$$\ell : \begin{cases} x = 2t \\ y = -t \\ z = 3t \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

T_2 reflects every vector in $\mathbb{R}^3$ about the z-axis, and T_3 reflects every vector in $\mathbb{R}^3$ about the x-axis.

a) Find the standard matrix of the composite transformation $T_3 \circ T_2 \circ T_1$.

[**Answer:** The standard matrix is $\dfrac{1}{14}\begin{pmatrix} -4 & 2 & -6 \\ -2 & 1 & -3 \\ -6 & 3 & -9 \end{pmatrix}$.]

b) Find the standard matrix of the composite transformation $T_1 \circ T_2 \circ T_3$.

[**Answer:** The standard matrix is $\frac{1}{14}\begin{pmatrix} -4 & -2 & -6 \\ 2 & 1 & 3 \\ -6 & -3 & -9 \end{pmatrix}$.]

c) Find the range of the composite transformation $T = T_3 \circ T_2 \circ T_1$.

[**Answer:** The range is given by the set span $\{\mathbf{u}\}$, where $\mathbf{u} = (2, 1, 3)$. That is, all the vectors lying on the line ℓ^* given by the following parametric equation:

$$\ell^* : \begin{cases} x = 2s \\ y = s \\ z = 3s \qquad \text{for all } s \in \mathbb{R}. \end{cases}$$]

11. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ for which the following is valid:

$$T(\mathbf{e_1}) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad T(\mathbf{e_2}) = \begin{pmatrix} -2 \\ k \\ 0 \end{pmatrix}, \quad T(\mathbf{e_3}) = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix},$$

where k is an unspecified real parameter and $\{\mathbf{e_1},\ \mathbf{e_2},\ \mathbf{e_3}\}$ are the standard basis vectors for $\mathbb{R}^3$.

a) Give the standard matrix of T and find $T(\mathbf{x})$, where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

[**Answer:** The standard matrix is $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & k & 1 \\ 1 & 0 & k \end{pmatrix}$. Then $T(\mathbf{x}) = \begin{pmatrix} 0 \\ 2k+3 \\ 3k+1 \end{pmatrix}$.]

b) Find all values of k, such that T is an injective transformation on $\mathbb{R}^3$.

[**Answer:** T is injective (one-to-one) on $\mathbb{R}^3$ for all $k \in \mathbb{R}\backslash\{-1,\ 2\}$.]

12. Consider a linear transformation T which projects every vector in $\mathbb{R}^3$ orthogonally onto the line of intersection of the following three planes:

$$\begin{aligned} \Pi_1 &: \quad x + 3y - 5z = 0 \\ \Pi_2 &: \quad x + 4y - 8z = 0 \\ \Pi_3 &: -2x - 7y + 13z = 0. \end{aligned}$$

a) Find the standard matrix A of T.

[**Answer:** $A = \frac{1}{13}\begin{pmatrix} 8 & -6 & -2 \\ -6 & 9/2 & 3/2 \\ -2 & 3/2 & 1/2 \end{pmatrix}$.]

b) Is this transformation T invertible? Explain.

[**Answer:** T is not invertible, since $\det A = 0$.]

13. Consider a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ for which the following is valid:

$$T\colon \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad T\colon \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad T\colon \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

a) Find the standard matrix A of T.

[**Answer:** $A = \begin{pmatrix} -8 & 14 & -3 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$.]

b) Is T an invertible transformation? Explain.

[**Answer:** The transformation T is invertible, as its standard matrix A is an invertible matrix.]

14. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, with standard matrix

$$A = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{pmatrix},$$

where α is an unspecified real parameter.

a) Find all values of α, such that T is injective on $\mathbb{R}^3$, as well as all the values of α, such that T is invertible.

[**Answer:** T is injective and invertible on $\mathbb{R}^3$ for all $\alpha \in \mathbb{R}\backslash\{1,\ -2\}$.]

b) Find the range R_T of T for $\alpha = -2$ and express the range in terms of a spanning set.

$$[\textbf{Answer:} \quad R_T = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.]$$

c) Let $\alpha = -2$ and find all $\mathbf{x} \in \mathbb{R}^3$, such that

$$T(\mathbf{x}) = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}.$$

$$[\textbf{Answer:} \quad \mathbf{x} = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{for all } t \in \mathbb{R}.]$$

15. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, where T reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about the plane $\Pi:\ ax + by + cz = 0$.

    a) Find the standard matrix A for T.

    [**Answer:** $A = \dfrac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}$.]

    b) Find a parametric equation of the line ℓ^*, such that ℓ^* is the reflection of the line

    $$\ell : \begin{cases} x = 2t + 1 \\ y = -3t \\ z = 2 \quad \text{for all } t \in \mathbb{R} \end{cases}$$

    about the plane $x + y - z = 0$.

    [**Answer:**

    $$\ell^* : \begin{cases} x = 8t + \dfrac{5}{3} \\ y = -7t + \dfrac{2}{3} \\ z = -2t + \dfrac{4}{3} \quad \text{for all } t \in \mathbb{R}. \end{cases}$$]

    c) Find the equation of the sphere which is a reflection of the sphere

    $$(x-1)^2 + (y+2)^2 + (z-1)^2 = 4$$

    about the plane $x + y - z = 0$.

    [**Answer:** $\left(x - \dfrac{7}{3}\right)^2 + \left(y - \dfrac{2}{3}\right)^2 + \left(z - \dfrac{1}{3}\right)^2 = 4.$]

16. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, where T reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about the line

    $$\ell : \begin{cases} x = at \\ y = bt \\ z = ct \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

    Find the standard matrix for T.

[**Answer:** $A = \dfrac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & -a^2+b^2-c^2 & 2bc \\ 2ac & 2bc & -a^2-b^2+c^2 \end{pmatrix}$.]

17. Assume that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that reflects every vector $\mathbf{x} \in \mathbb{R}^3$ about a line ℓ, such that T has the following standard matrix:

$$A = \frac{1}{7}\begin{pmatrix} -3 & -2 & 6 \\ -2 & -6 & -3 \\ 6 & -3 & 2 \end{pmatrix}.$$

Find a parametric equation for this line ℓ.

[**Answer:**

$$\ell : \begin{cases} x = t \\ y = -\dfrac{1}{2}t \\ z = \dfrac{3}{2}t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$]

18. Consider three linear transformations, T_1, T_2 and T_3, that map all vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$ as follows:
T_1 rotates every vector counter-clockwise by angle $\theta_1 = \pi$ about the z-axis;
T_2 rotates every vector counter-clockwise by angle θ_2 about the y-axis;
T_3 rotates every vector counter-clockwise by angle θ_3 about the x-axis.

a) Find the standard matrix A for the composite transformation $T = T_1 \circ T_2 \circ T_3$.

[**Answer:**

$$A = \begin{pmatrix} -\cos\theta_2 & \sin\theta_2 \sin\theta_3 & \cos\theta_3 \sin\theta_2 \\ 0 & -\cos\theta_3 & \sin\theta_3 \\ \sin\theta_2 & \cos\theta_2 \sin\theta_3 & \cos\theta_2 \cos\theta_3 \end{pmatrix}.$$]

b) Find θ_2 and θ_3 with $0 \le \theta_2 \le \pi$ and $0 \le \theta_3 \le \pi$, such that

$$T\colon (1,2,0) \mapsto (0,-1,2),$$

where T is the transformation in a) above. To which point does $(0,-1,2)$ map under T for those values of θ_2 and θ_3. That is find $T(0,-1,2)$.

[**Answer:** $\theta_2 = \frac{\pi}{6}$, $\theta_3 = \frac{\pi}{3}$ and $T(0,-1,2) = (\frac{1}{2} - \frac{\sqrt{3}}{4},\ \frac{1}{2} + \sqrt{3},\ \frac{3}{4} + \frac{\sqrt{3}}{2})$.]

c) Show that T as obtained in part b), is an injective transformation on $\mathbb{R}^3$ by calculating the determinant of its standard matrix A and find the standard matrix for the inverse transformation of T, i.e the standard matrix for T^{-1}. Show also that T^{-1} maps the point $(0,-1,2)$ back to the point $(1,2,0)$.

[**Answer:** Since $\det A \neq 0$, the matrix A is invertible, which means that T is invertible and the standard matrix of T^{-1} is A^{-1}, namely

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -2\sqrt{3} & 0 & 2 \\ \sqrt{3} & -2 & 3 \\ 1 & 2\sqrt{3} & \sqrt{3} \end{pmatrix}.\ ]$$

d) Consider the line ℓ, given by the parametric equation

$$\ell : \begin{cases} x = -2t + 1 \\ y = 3t - 2 \\ z = t + 4 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Make use of the linear transformation T obtained in part b) and find a parametric equation of the line ℓ^*, such that ℓ^* is the image of the line ℓ under T. That is find ℓ^*, such that

$$T\colon \ell \to \ell^*.$$

[**Answer:**

$$\ell^* : \begin{cases} x = \left(\frac{7\sqrt{3}}{4} + \frac{1}{4}\right) t - \sqrt{3} + 1 \\ y = \left(\frac{\sqrt{3}}{2} - \frac{3}{2}\right) t + 2\sqrt{3} + 1 \\ z = \left(\frac{\sqrt{3}}{4} + \frac{5}{4}\right) t + \sqrt{3} - 1 \quad \text{for all } t \in \mathbb{R}.\ ] \end{cases}$$

e) Consider the plane

$$\Pi: \; 2x - 3y + z = 4.$$

Make use of the linear transformation T obtained in part b) and find the equation of the plane Π^*, such that Π^* is the image of the plane Π under T. That is find Π^*, such that

$$T: \Pi \to \Pi^*.$$

Use Maple to sketch both Π and Π^* (see Appendix A for information about Maple).

[**Answer:** $\Pi^*: \; (7\sqrt{3} - 1)x - 2(\sqrt{3} + 3)y - (\sqrt{3} - 5)z = -16.$]

19. Consider four linear transformations, T_1, T_2, T_3 and T_4, that map all vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$ as follows: T_1 rotates every vector counter-clockwise by angle π about the z-axis;
T_2 rotates every vector counter-clockwise by angle $\pi/3$ about the y-axis;
T_3 rotates every vector counter-clockwise by angle $-\pi/2$ about the x-axis;
T_4 rotates every vector counter-clockwise by angle $\pi/2$ about the z-axis.

a) Find the standard matrix A of the transformation $T = T_4 \circ T_2 \circ T_1 \circ T_3$.

[**Answer:**

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix}.\ ]$$

b) Find the standard matrix of the inverse transformation of T, where T is the transformation in a) above.

[**Answer:** The standard matrix of T^{-1} is given by A^{-1}, where A is the standard matrix of T obtained in a) above. That is

$$A^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}.\ ]$$

c) Find a parametric equation for the line ℓ, such that

$$T \colon \ell \to \ell^*,$$

where T is the the transformation obtained in a) above and ℓ^* is

$$\ell^* : \begin{cases} x = 6t - 1 \\ y = -2t + 2 \\ z = 2t \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

[**Answer:**

$$\ell : \begin{cases} x = (-\sqrt{3} + 1)s - 1 \\ y = -(\sqrt{3} + 1)s + \sqrt{3} \\ z = 6s - 1 \quad \text{for all } s \in \mathbb{R}.\ ] \end{cases}$$

20. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix A. Consider further two linearly independent vectors,

$$\mathbf{u} = (u_1, u_2), \qquad \mathbf{v} = (v_1, v_2),$$

which describe a paralleleogram in $\mathbb{R}^2$ with area S. The transforamtion T then maps area S to area $T(S)$. Show that

$$\text{area } T(S) = |\det A| \; (\text{area } S).$$

21. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix A. Consider further three linearly independent vectors,

$$\mathbf{u} = (u_1, u_2, u_3), \qquad \mathbf{v} = (v_1, v_2, v_3), \qquad \mathbf{w} = (w_1, w_2, w_3)$$

which describe a parallelepiped in $\mathbb{R}^3$ with volume V. The transforamtion T then maps volume V to volume $T(V)$. Show that

$$\text{volume } T(V) = |\det A| \; (\text{volume } V).$$

Appendix A

Matrix calculations with Maple

Maple is a commercial computer algebra system developed and sold commercially by *Maplesoft*, a software company based in Waterloo, Canada. It was first developed in 1980 by the *Symbolic Computation Group* at the University of Waterloo. The Maple system is written in the programming languages *C* and *Java*. In this appendix we describe a few main *Maple* commands for performing some of the basic vector and matrix calculations.

For any vector or matrix calculation, we first need to load the package *LinearAlgebra*. This is done by writing

with(LinearAlgebra)

in the beginning of a *Worksheet Mode* file on the command line, i.e. after the symbol

$>$

It is a good idea to always start your new Worksheet file with a *restart*, in order to clear all possible assigned values and parameters when the file is compiled. That is, we write on the first line

$>$ *restart*

Below, we show how to assign an $\mathbb{R}^3$ vector $\mathbf{u}$ and a 2×3 matrix A.

$> with(LinearAlgebra):$

$> u :=< a, b, c >$

$$u := \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$> A :=< a, b|c, d|e, f >$

$$A := \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Alternatively, we may also define the same vector **u** and matrix A in the following manner:

$> with(LinearAlgebra):$

$> u := Vector([a, b, c])$

$$u := \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$> A := Matrix(2, 3, [a, b, c, d, e, f])$

$$A := \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Note that the sentence that follows the sign ♯ is a comment. Note further that selected **help** and examples are available for a particular Maple routine or function by pointing the curser on a word in the Maple Worksheet, e.g. *Matrix*, followed by hitting **F2** on the keyboard.

We now show how to perform some basic vector and matrix calculations

$> u :=< 4, -1, -1 >$ $\quad \sharp$*Vector* **u** *with coordinates* $(4, -1, -1)$ *is defined.*

$$u := \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$> v :=< 1, 0, 1 >$ $\quad \sharp$*Vector* **v** *with coordinates* $(1, 0, 1)$ *is defined.*

$$v := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$> u + (-v)$ $\quad \sharp$*The sum* $\mathbf{u} + (-\mathbf{v})$.

$$\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

$> DotProduct(u, v)$ $\quad \sharp$*The dot product between* **u** *and* **v**.

$> norm(u, 2)$ $\quad \sharp$*The norm of the vector* $\mathbf{u}$.

$$3\sqrt{2}$$

$> theta := arccos\left(\frac{u.v}{norm(u,2) \cdot norm(v,2)}\right)$ $\quad \sharp$*The angle* θ *between* $\mathbf{u}$ *and* $\mathbf{v}$.

$$\theta := \frac{1}{3}\pi$$

$> CrossProduct(u, v)$ $\quad \sharp$*The cross-product between* $\mathbf{u}$ *and* $\mathbf{v}$.

$$\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$$

$> A :=< 1, -3, 5 | 2, -4, 2 | -1, 2, 3 >$

$$A := \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

$> b :=< 1, 2, -3 >$

$$b := \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$> x := LinearSolve(A, b)$ $\quad \sharp$*The solution* $\mathbf{x}$ *of the system* $A\mathbf{x} = \mathbf{b}$.

$$x := \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

$> A.x - b$ $\quad \sharp$*Check the solution* $\mathbf{x}$ *of* $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$> AM := Matrix([A, b])$ $\quad \sharp$*The augmented matrix* $[A\ \mathbf{b}]$.

$$AM := \begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix}$$

$> \mathit{GaussianElimination}(AM)$ $\quad \sharp$*Perform Gauss elimination on* $[A\ \mathbf{b}]$.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 4 & 12 \end{bmatrix}$$

$> \mathit{ReducedRowEchelonForm}(AM)$ $\quad \sharp$*Reduced row echolon form of* $[A\ \mathbf{b}]$.

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$> \mathit{Determinant}(A)$ $\quad \sharp$*The determinant of* A.

$$8$$

$> \mathit{Ainv} := \mathit{MatrixInverse}(A)$ $\quad \sharp$*The inverse matrix of* A.

$$\mathit{Ainv} := \begin{bmatrix} -2 & -1 & 0 \\ \frac{19}{8} & 1 & \frac{1}{8} \\ \frac{7}{4} & 1 & \frac{1}{4} \end{bmatrix}$$

$> \mathit{Ainv}.A$ $\quad \sharp$*Calculate* $A^{-1}A$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$> x := \mathit{Ainv}.b$ $\quad \sharp$*The solution* $\mathbf{x} = A^{-1}\mathbf{b}$ *of the system* $A\mathbf{x} = \mathbf{b}$.

$$x := \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

$> B := < 1, -3, 5 | 2, -4, 2 | -1, 1, 3 >$ $\quad \sharp$*We consider another example.*

$$B := \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 1 \\ 5 & 2 & 3 \end{bmatrix}$$

$> c := < 0, -2, 8 >$

$$b := \begin{bmatrix} 0 \\ -2 \\ 8 \end{bmatrix}$$

$> Determinant(B)$ ♯*The determinant of B.*

$$0$$

$> Binv := MatrixInverse(B)$ ♯*B is singular.*

Error, (in *MatrixInverse*) singular matrix

$> x := LinearSolve(B, c)$ ♯*The solution* **x** *of* $B\mathbf{x} = \mathbf{c}$ *that contains an arbitrary parameter denoted by Maple as* $\_t0_3$.

$$x := \begin{bmatrix} 2 - \_t0_3 \\ -1 + \_t0_3 \\ \_t0_3 \end{bmatrix}$$

To plot figures in $\mathbb{R}^3$ we use the Maple function *plot3d*. Consider for example the plane

$$\frac{5}{7}x - \frac{19}{21}y - z = -\frac{4}{7}.$$

We plot this plane on the x-interval $[0,\ 8]$ and the y-interval $[0,\ 20]$. In order to plot this plane such that we can see the x-axis, the y-axis and the z-axis, we use the following Maple commands:

$$> plot3d\left(\left[\frac{5}{7}x - \frac{19}{21}y + \frac{4}{7}\right], x = 0..8, y = 0..20, axes = boxed\right)$$

See Figure A.1 for the output plot of this plane.

We now consider the following three planes:

$$\frac{5}{7}x - \frac{19}{21}y - z = -\frac{4}{7}$$

$$-\frac{19}{26}x - \frac{4}{13}y - z = -\frac{3}{26}$$

$$x + y - z = 0.$$

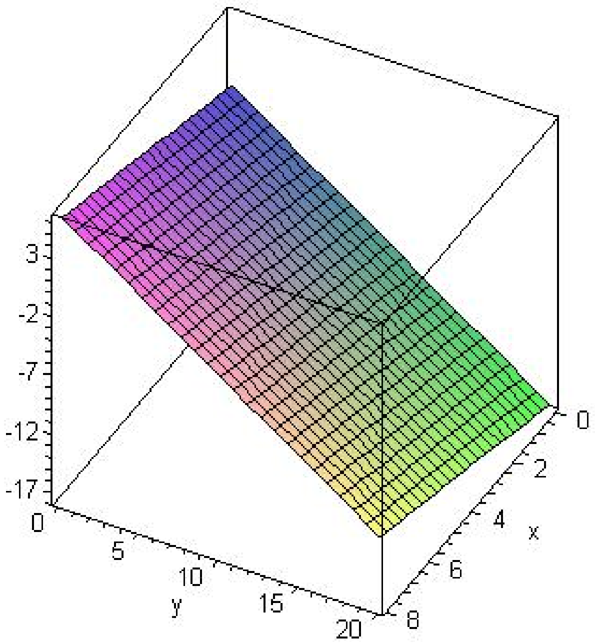

Figure A.1: The plot of a plane.

We calculate the intersection of the three given planes.

> *restart*

> *with(LinearAlgebra)*

> $A := < \frac{5}{7}, -\frac{19}{26}, 1 | -\frac{19}{21}, -\frac{4}{13}, 1 | -1, -1, -1 >$

$$A := \begin{bmatrix} \frac{5}{7} & -\frac{19}{21} & -1 \\ -\frac{19}{26} & -\frac{4}{13} & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

> $b := < -\frac{4}{7}, -\frac{3}{26}, 0 >$

$$b := \begin{bmatrix} -\frac{4}{7} \\ -\frac{3}{26} \\ 0 \end{bmatrix}$$

$> x := LinearSolve(A, b) \quad \sharp \textit{The solution } \mathbf{x} \textit{ of } A\mathbf{x} = \mathbf{b}$

$$x := \begin{bmatrix} -\frac{24}{133} \\ \frac{87}{266} \\ \frac{39}{266} \end{bmatrix}$$

We conclude the the three planes intersect in the point $(-\frac{24}{133}, \frac{87}{266}, \frac{39}{266})$.

We now plot the intersection of the above given three planes on the x-interval $[-8, 8]$ and the y-interval $[-20, 20]$. Note the command *plotlist=true*, which is necessary when plotting more than two planes on the same graph. The colon (:) at the end of an input line hides the Maple output.

$> P1 := \frac{5}{7}x - \frac{19}{21}y + \frac{4}{7} : \quad P2 := -\frac{19}{26}x - \frac{4}{13}y + \frac{3}{26} : \quad P3 := x + y :$

$> plot3d\left([P1, P2, P3], x = -8..8, y = -20..20, plotlist = true, color = [blue, red, green]\right)$

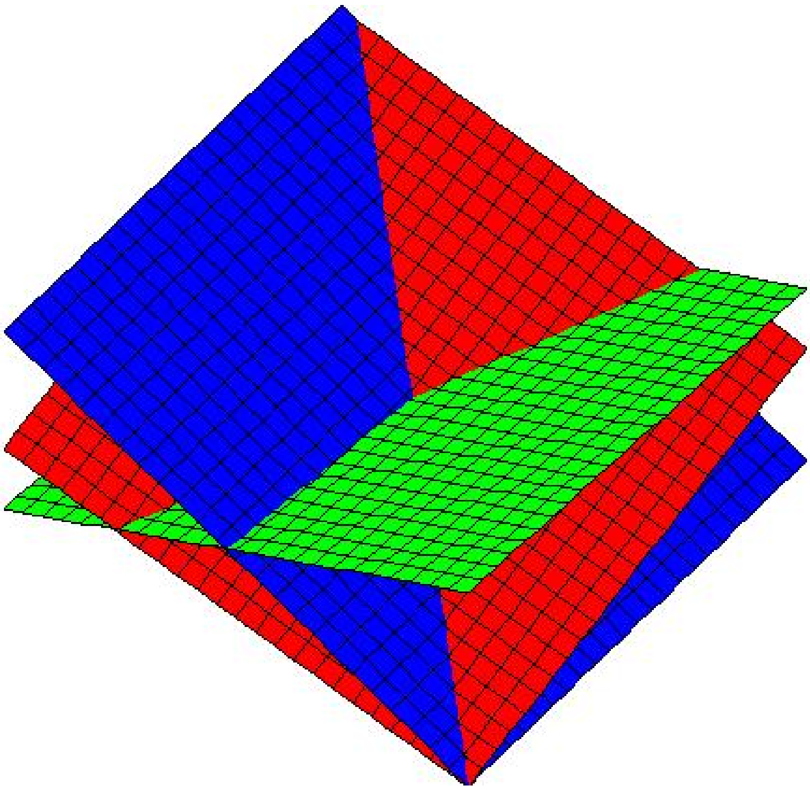

Figure A.2: The intersection of three planes in a common point.

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