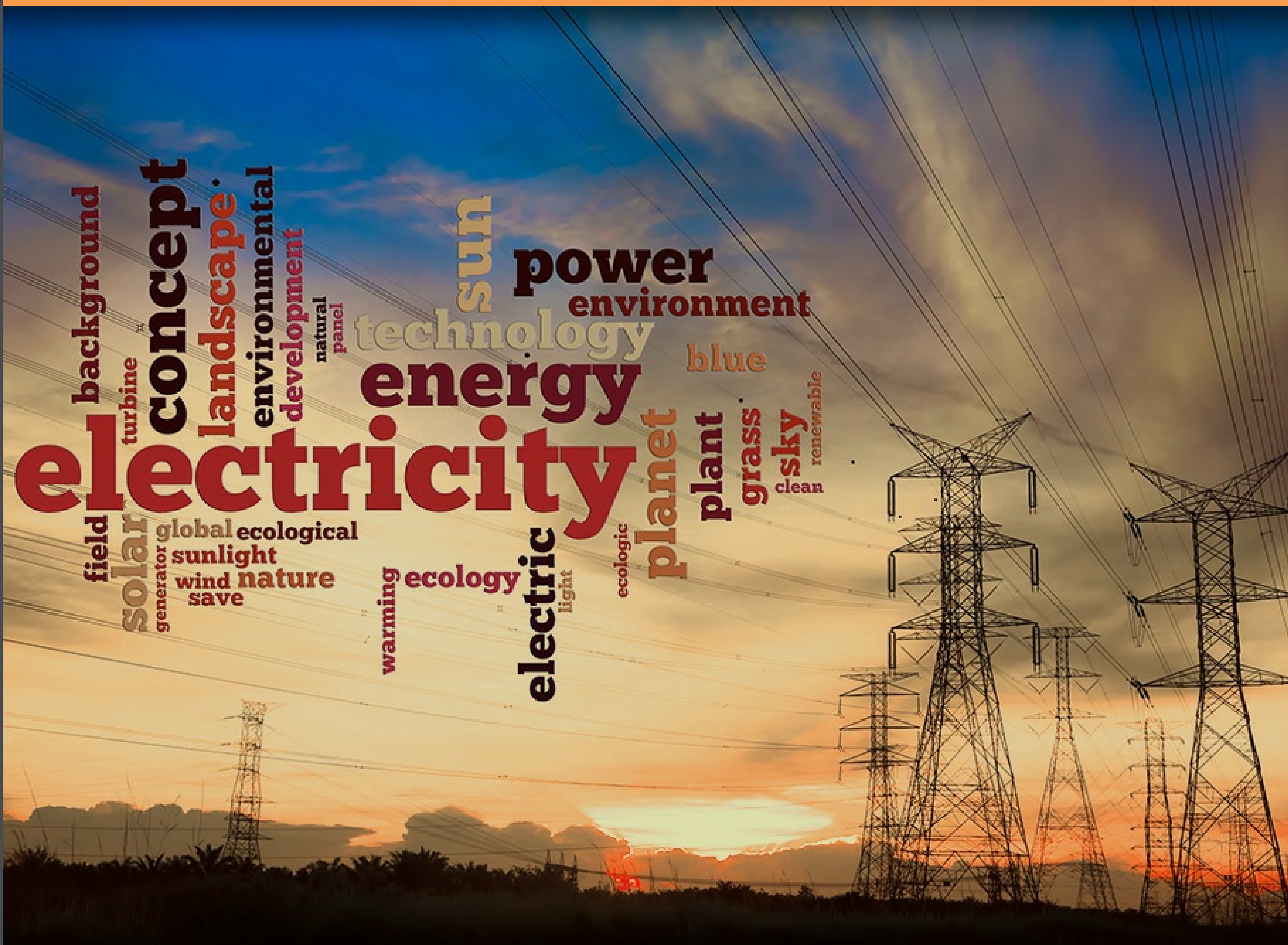
Essential Electrodynamics: Solutions

Raymond John Protheroe



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Solutions

Essential Electrodynamics: Solutions
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Preface

This book gives the solutions to the exercises at the end of each chapter of my book "Essential Electrodynamics" (also published by Ventus Publishing ApS). I recommend that you attempt a particular exercise after reading the relevant chapter, and before looking at the solutions published here.

Often there is more than one way to solve a problem, and obviously one should use any valid method that gets the result with the least effort. Usually this means looking for symmetry in the problem – for example from the information given can we say that from symmetry arguments the field we need to derive can only be pointing in a certain direction. If so, we only need to calculate the component of the field in that direction, or we may be able to use Gauss' law or Ampère's law to enable us to write down the result. In some of these exercise solutions the simplest route to the solution is deliberately not taken in order to illustrate other methods of solving a problem, but in these cases the simpler method is pointed out.

The solutions to the exercise problems for each chapter of "Essential Electrodynamics" are presented here in the corresponding chapters of "Essential Electrodynamics - Solutions".

I hope you find these exercises useful. If you find typos or errors I would appreciate you letting me know. Suggestions for improvement are also welcome – please email them to me at protheroe.essentialphysics@gmail.com.

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Adelaide, May 2013

1 Electrodynamics and conservation laws

1–1 A magnetic dipole of moment $\mathbf{m} = m\widehat{\mathbf{z}}$ is located at the origin. A thin circular conducting ring of radius a vibrates such that the position of its centre is $\mathbf{r} = [z_0 + b\cos(\omega t)]\widehat{\mathbf{z}}$ with $b \ll a \ll z_0$. The plane of the ring remains parallel to the x—y plane during the vibration. Find the emf around the ring in the ϕ direction.

Solution

The magnetic field of the dipole is

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r}) - r^2\mathbf{m}}{r^5}\right]. \tag{1.1}$$

Since $b \ll a \ll z_0$ we can approximate the magnetic field anywhere on the ring as it vibrates by

$$\mathbf{B}[z_r(t)\,\widehat{\mathbf{z}})] \;=\; \frac{\mu_0}{4\pi}\left[\frac{3z_r\,\widehat{\mathbf{z}}\,(m\,\widehat{\mathbf{z}}\cdot z_r\,\widehat{\mathbf{z}}) - z_r^2 m\,\widehat{\mathbf{z}}}{z_r^5}\right] \;=\; \frac{\mu_0}{2\pi}\frac{m\,\widehat{\mathbf{z}}}{z_r^3}, \tag{1.2}$$

where $z_r(t) = [z_0 + b\cos(\omega t)]$ is the height of the ring.

The magnetic flux through the loop is

$$\begin{aligned}
\Phi_B(t) \;&=\; \pi a^2\frac{\mu_0 m}{2\pi}\,[z_0 + b\cos(\omega t)]^{-3}, & (1.3)\\
&=\; \frac{\mu_0 m a^2}{2z_0^3}\left[1 + \frac{b}{z_0}\cos(\omega t)\right]^{-3}, & (1.4)\\
&\approx\; \frac{\mu_0 m a^2}{2z_0^3}\left[1 - 3\frac{b}{z_0}\cos(\omega t)\right] & (1.5)
\end{aligned}$$

since $a \ll z_0$. Hence,

$$\begin{aligned}
\mathcal{E} \;&=\; -\frac{d\Phi_B}{dt}, & (1.6)\\
\therefore\;\; \mathcal{E} \;&\approx\; -\frac{3\mu_0\, m\, a^2 b\,\omega}{2z_0^4}\sin(\omega t). & (1.7)
\end{aligned}$$

1–2 A thin disc of radius a and height h contains charge $+q$ uniformly distributed throughout the disc. The disc is located with its centre at the origin, and rotates about the z-axis with angular velocity $\boldsymbol{\omega} = \omega\widehat{\mathbf{z}}$.

(a) Using cylindrical coordinates but with R being the cylindrical radius to avoid confusion with the charge density $\rho(\mathbf{r})$, specify the current density $\mathbf{J}(R, \phi, z)$ as a function of position. In the limit $h \ll a$ find the magnetic dipole moment.
(b) Consider a circular loop of radius R_0 around the z-axis at height z_0 above the disc for the case $R_0 \ll a \ll z_0$. Find the magnetic flux through the loop, and hence find the vector potential at the loop.
(c) If, due to friction in the axle, the disc's angular velocity is decreasing exponentially with time t as $\omega(t) = \omega_0 e^{-t/t_0}$, where t_0 is the decay time scale, find the electric field at the loop at time $t = 0$.

Solution

(a) Within the disc, $\rho(\mathbf{r}) = \dfrac{q}{\pi a^2 h}$ and $\mathbf{v}(\mathbf{r}) = R\omega\widehat{\boldsymbol{\phi}}$, and so

$$\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\,\mathbf{v}(\mathbf{r}) \;=\; \rho(\mathbf{r})R\omega\widehat{\boldsymbol{\phi}} \;=\; \frac{qR\omega}{\pi a^2 h}\widehat{\boldsymbol{\phi}}. \tag{1.8}$$

The dipole moment is

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\int \mathbf{r} \times \mathbf{J}(\mathbf{r})\, d^3r, && (1.9)\\
&= \frac{h}{2}\int_0^a (R\widehat{\mathbf{R}}) \times \left(\frac{qR\omega}{\pi a^2 h}\widehat{\boldsymbol{\phi}}\right) 2\pi R\, dR, && (1.10)\\
&= \frac{q\omega}{a^2}\int_0^a R^3\, dR\, \widehat{\mathbf{z}}, && (1.11)\\
&= \frac{q\omega a^2}{4}\,\widehat{\mathbf{z}}. && (1.12)
\end{aligned}$$

(b) The circular loop is close to the axis of the dipole, but a distance $z_0 \gg a$ away. The magnetic field of a dipole is

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}\cdot\widehat{\mathbf{r}})\,\widehat{\mathbf{r}} - \mathbf{m}]. \tag{1.13}$$

$$\therefore\;\; \mathbf{B}(0,0,z_0) \;=\; \frac{\mu_0}{4\pi z_0^3}2m\,\widehat{\mathbf{z}} \;=\; \frac{\mu_0}{8\pi z_0^3}q\omega a^2\,\widehat{\mathbf{z}}. \tag{1.14}$$

The magnetic flux through the loop is then approximately

$$\Phi_B \;=\; \pi R_0^2\, B(0,0,z_0) \;=\; \frac{\mu_0}{8}\frac{R_0^2}{z_0^3}q\omega a^2. \tag{1.15}$$

We can obtain the vector potential from the magnetic flux using

$$\oint_\Gamma \mathbf{A}\cdot d\mathbf{r} \;=\; \int_S \mathbf{B}\cdot d\mathbf{S} \;=\; \Phi_B, \tag{1.16}$$

where loop Γ is the circular loop and S is any surface bounded by the loop. From symmetry arguments the vector potential must be in the $\widehat{\boldsymbol{\phi}}$ direction

$$\therefore\;\; \mathbf{A}(R_0,\phi,z_0) \;=\; \frac{\Phi_B}{2\pi R_0}\,\widehat{\boldsymbol{\phi}} \;=\; \frac{\mu_0}{16\pi}\frac{a^2 R_0}{z_0^3} q\omega\,\widehat{\boldsymbol{\phi}}. \tag{1.17}$$

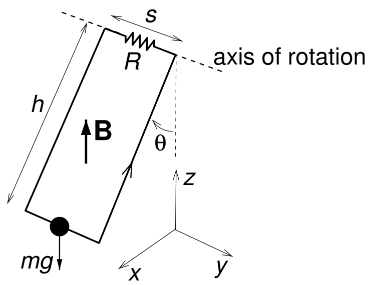
(c) If $\omega(t) \;=\; \omega_0 e^{-t/t_0}$ then

$$\mathbf{E}(R_0,\phi,z_0,t) \;=\; -\frac{\partial}{\partial t}\mathbf{A}(R_0,\phi,z_0,t), \tag{1.18}$$

$$\;=\; -\frac{\mu_0}{16\pi}\frac{a^2 R_0}{z_0^3} q\omega_0 \frac{\partial}{\partial t} e^{-t/t_0}\widehat{\boldsymbol{\phi}}, \tag{1.19}$$

$$\;=\; \frac{\mu_0}{16\pi}\frac{a^2 R_0}{z_0^3}\frac{q\omega_0}{t_0} e^{-t/t_0}\widehat{\boldsymbol{\phi}}. \tag{1.20}$$

1–3 A light rigid rectangular circuit with resistance R has mass m attached to the middle of the lower side (width s), and the top side is suspended horizontally using frictionless bearings to form a simple pendulum of length h as shown in the diagram below. In the absence of a magnetic field the position of the pendulum mass would be described by $\mathbf{r}_m(t) \approx h\,\theta_0 \cos(\omega t)\,\widehat{\mathbf{x}}$ where $\omega = \sqrt{g/h}$. A uniform magnetic field $\mathbf{B}$ points in the vertically upward direction.



(a) Assuming the position of the pendulum mass is still described by $\mathbf{r}_m(t) \approx h\,\theta_0 \cos(\omega t)\,\widehat{\mathbf{x}}$, what is the magnetic flux $\Phi_B(t)$ through the circuit, and hence the emf as a function of time? Take the direction around the circuit indicated by the arrow to correspond to positive emfs and currents.

(b) What is the force on the lower side of the circuit due to the magnetic field? What is the instantaneous work done by the pendulum *against* this force? Compare this with instantaneous power dissipated in the circuit? What are the consequences of the presence of the magnetic field for the motion of the pendulum?

Solution

(a) The magnetic flux through the circuit is

$$\Phi_B(t) \quad = \quad \widehat{\mathbf{x}} \cdot \mathbf{r}_m(t)\, s\, B \quad = \quad h\theta_0 \cos(\omega t) s B \tag{1.21}$$

an so the emf is

$$\mathcal{E}(t) \quad = \quad -\frac{d\Phi_B}{dt} \quad = \quad \omega h s B \theta_0 \sin(\omega t). \tag{1.22}$$

The current is

$$I(t) \quad = \quad \mathcal{E}(t)/R \quad = \quad (\omega h s B \theta_0 / R) \sin(\omega t). \tag{1.23}$$

(b) The Lorentz force on the lower side of the circuit is

$$\begin{aligned}
\mathbf{F}(t) &= I\int_{\text{lower side}} d\mathbf{r}' \times \mathbf{B}, && (1.24)\\
&= (\omega hsB\theta_0/R)\sin(\omega t)\,(s\,\widehat{\mathbf{y}}) \times (B\,\widehat{\mathbf{z}}), && (1.25)\\
&= (\omega hs^2B^2\theta_0/R)\sin(\omega t)\,\widehat{\mathbf{x}}, && (1.26)
\end{aligned}$$

and since $\mathbf{v}(t) = d\mathbf{r}/dt = -\omega h\theta_0 \sin(\omega t)\,\widehat{\mathbf{x}}$, we note that $\mathbf{F}(t)$ is in the opposite direction to $\mathbf{v}(t)$. The instantaneous work done by the pendulum *against* this force is

$$\begin{aligned}
P_{\text{mech}}(t) &= -\mathbf{F}(t)\cdot\mathbf{v}(t), && (1.27)\\
&= -[(\omega hs^2B^2\theta_0/R)\sin(\omega t)\,\widehat{\mathbf{x}}]\cdot[-\omega h\,\theta_0\sin(\omega t)\,\widehat{\mathbf{x}}], && (1.28)\\
&= (\omega^2h^2s^2B^2\theta_0^2/R)\sin^2(\omega t). && (1.29)
\end{aligned}$$

Note that The instantaneous power dissipated as heat in the resistor is

$$P_{\text{heat}}(t) = \frac{\mathcal{E}^2}{R} = \frac{[\omega hsB\theta_0\sin(\omega t)]^2}{R} \tag{1.30}$$

consistent with conversion of the mechanical energy of the pendulum to heat.

(c) The pendulum's amplitude θ_0 will decay over time. To determine the rate of decay, we can consider first consider the average rate of energy loss of the pendulum which is equal to the time-average of the power dissipated as heat

$$\langle P_{\text{heat}}\rangle = \frac{(\omega hsB\theta_0)^2}{2R} \tag{1.31}$$

and compare this with the total energy of the pendulum which is equal to its kinetic energy at $x = 0$

$$W_{\text{tot}} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m(\omega h\theta_0)^2. \tag{1.32}$$

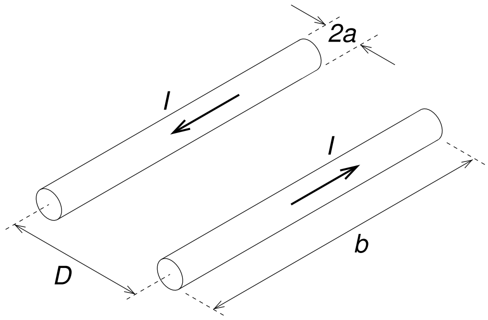
The energy will decay exponentially with timescale

$$\tau_{\text{energy}} = \frac{W_{\text{tot}}}{\langle P_{\text{heat}}\rangle} = \frac{1}{2}m(\omega h\theta_0)^2\frac{2R}{(\omega hsB\theta_0)^2} = \frac{mR}{s^2B^2}, \tag{1.33}$$

that is, $W_{\text{tot}}(t) \propto e^{-t/\tau_{\text{energy}}}$. Since the energy is proportional to θ_0^2 the pendulum amplitude θ_0 will decay exponentially as $\theta_0(t) \propto e^{-t/\tau_\theta}$ where $\tau_\theta = 2\tau_{\text{energy}}$.

1–4 Consider the section of a two-wire transmission line shown below. Show that the self-inductance per unit length for the case where $D \gg a$ is given by

$$L = \frac{\mu_0}{\pi} \ln \frac{D}{a}. \tag{1.34}$$



Solution

We need to calculate the magnetic flux between the two wires for a section of length b due to current I flowing in both wires. For just one wire the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \tag{1.35}$$

and integrating between the conductors we get the flux for length b

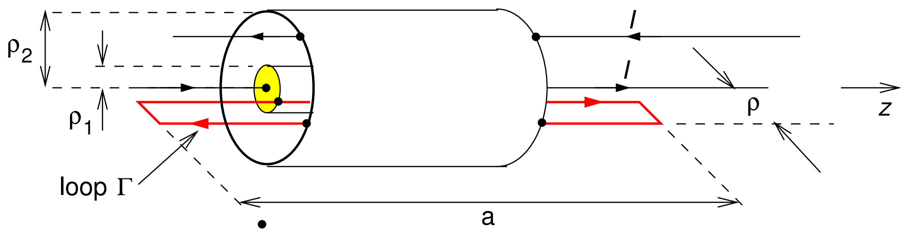
$$\Phi_B = b\frac{\mu_0 I}{2\pi}\int_a^{D-a}\frac{d\rho}{\rho} = b\frac{\mu_0 I}{2\pi}\ln\left(\frac{D-a}{a}\right) \tag{1.36}$$

Multiplying by 2 (to include the flux from both wires) and dividing by b and the current we get the inductance per unit length

$$L = \frac{\mu_0}{\pi}\ln\left(\frac{D-a}{a}\right) = \frac{\mu_0}{\pi}\ln\left(\frac{D}{a}\right) \quad (D \gg a). \tag{1.37}$$

1–5 Consider a coaxial cable as an infinite cylindrical inductor and find the inductance per unit length.

Solution



Consider length a with current I in the $\hat{\mathbf{z}}$ direction on the inner conductor and I in the $-\hat{\mathbf{z}}$ direction on the outer conductor. To find the magnetic field, use Ampere's law for a coaxial circular loop of radius ρ, for $\rho_1 < \rho < \rho_2$,

$$2\pi\rho\, B(\rho,\phi,z) = \mu I. \qquad \therefore \ \mathbf{B}(\rho,\phi,z) = \frac{\mu I}{2\pi\rho}\hat{\boldsymbol{\phi}} \quad (\rho_1 < \rho < \rho_2). \tag{1.38}$$

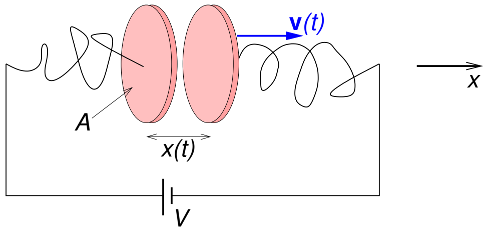
The magnetic flux through the rectangular loop Γ of length a and width $(\rho_2 - \rho_1)$ spanning radii $\rho_1 < \rho < \rho_2$ as shown is

$$\Phi_B = \int_{\rho_1}^{\rho_2}\frac{\mu I}{2\pi\rho}a\,d\rho = \frac{\mu a I}{2\pi}\ln\left(\frac{\rho_2}{\rho_1}\right). \tag{1.39}$$

Hence, the inductance per unit length is

$$L = \frac{\Phi_B}{I\,a} = \frac{\mu}{2\pi}\ln\left(\frac{\rho_2}{\rho_1}\right). \tag{1.40}$$

1–6 The diagram shows a parallel plate capacitor. Find the current $I(t)$, and the displacement current density $\mathbf{J}_D$ between moving capacitor plates, and check that the total displacement current is $I_D(t) = I(t)$. Neglect fringing effects.



Solution

From the capacitance of a parallel plate capacitor we get the charge on the positive plate

$$Q(t) = C(t)V = \frac{\varepsilon_0 A}{x(t)}V. \tag{1.41}$$

So there is a current flowing

$$I(t) = \frac{dQ}{dt} = -\frac{\varepsilon_0 AV}{[x(t)]^2}\frac{dx}{dt} = -\frac{\varepsilon_0 AV}{[x(t)]^2}v(t). \tag{1.42}$$

The electric field between the plates is

$$\mathbf{E}(t) = \frac{V}{x(t)}\widehat{\mathbf{x}}. \tag{1.43}$$

The displacement current density between the plates is then

$$\mathbf{J}_D = \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\varepsilon_0\frac{V}{[x(t)]^2}\frac{dx}{dt}\widehat{\mathbf{x}} = -\varepsilon_0\frac{V}{[x(t)]^2}v(t)\widehat{\mathbf{x}}. \tag{1.44}$$

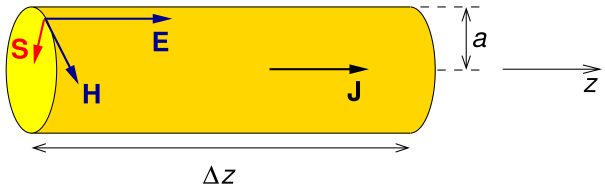
Multiply $\mathbf{J}_D$ by the plate area $A\widehat{\mathbf{z}}$ to get the total displacement current

$$I_D = \mathbf{J}_D \cdot A\widehat{\mathbf{x}} = -\varepsilon_0\frac{V}{[x(t)]^2}Av(t) \tag{1.45}$$

which is the same as the current I in the wire.

1–7 Consider a straight piece of wire radius a and length Δz, along which current I is flowing. The potential difference between the ends is ΔV. Find the Poynting vector at the surface of the wire and use it to determine the rate at which energy flows into the wire, and compare the result with Joules's law.

solution



When a current flows along a resistive wire, work is done which shows up as Joule heating. The energy flows from the electromagnetic field into the wire, and the rate at which energy is flowing can be determined from the Poynting vector. If the wire has potential difference ΔV between its ends, the electric field is parallel to the wire and given by

$$\mathbf{E} = \frac{\Delta V}{\Delta z}\widehat{\mathbf{z}} \tag{1.46}$$

From Ampere's law the magnetic intensity of current I is circumferential, and at the surface has the value

$$\mathbf{H} = \frac{I}{2\pi a}\widehat{\boldsymbol{\phi}}. \tag{1.47}$$

Then, the Poynting vector is

$$\mathbf{S} \;=\; \left(\frac{\Delta V}{\Delta z}\widehat{\mathbf{z}}\right) \times \left(\frac{I}{2\pi a}\widehat{\boldsymbol{\phi}}\right) \;=\; -\frac{\Delta V\, I}{2\pi a\, \Delta z}\widehat{\boldsymbol{\rho}}. \tag{1.48}$$

The Poynting vector points radially inward and so the energy per unit time passing in through the surface of length Δz of the wire, which has area $2\pi\, a\, \Delta z$, is

$$-\int_{\Sigma} \mathbf{S} \cdot d\boldsymbol{\Sigma} \;=\; \frac{\Delta V\, I}{2\pi\, a\, \Delta z} 2\pi\, a\, \Delta z = \Delta V I, \tag{1.49}$$

where $d\boldsymbol{\Sigma}$ is used here for the surface element to avoid confusion with the Poynting vector $\mathbf{S}$. The current density has magnitude $J = I/\pi a^2$, so the Joule heating power of the wire, which has volume $\pi a^2 \Delta z$, is

$$\int_{\text{vol}} \mathbf{E} \cdot \mathbf{J}\, d^3 r \;=\; \left(\frac{\Delta V}{\Delta z}\right)\left(\frac{I}{\pi a^2}\right)\left(\pi a^2 \Delta z\right) \;=\; \Delta V I. \tag{1.50}$$

The current and voltage are constant, and so the magnetic and electric fields also are constant,

$$-\frac{d}{dt}\int_{\text{vol}} \left(\frac{\varepsilon E^2}{2} + \frac{B^2}{2\mu}\right) d^3 r = 0. \tag{1.51}$$

The result, Poynting flux into the wire (Eq. 1.50) equals Joule heating power (Eq. 1.49) plus rate of increase of field energy inside wire (Eq. 1.51), is consistent with Poynting's theorem.

1–8 A long solenoid carrying a time-dependent current $I(t)$ is wound on a hollow cylinder whose axis of symmetry is the z-axis. The solenoid's radius is a, and it has n turns per metre.

(a) Write down the magnetic intensity $\mathbf{H}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ everywhere. What is the energy density in the magnetic field inside the solenoid?
(b) Find the electric field $\mathbf{E}(\mathbf{r},t)$ everywhere using Faraday's law in integral form.
(c) Find the magnetic vector potential $\mathbf{A}(\mathbf{r},t)$ everywhere.
(d) Find the Poynting vector $\mathbf{S}(\mathbf{r},t)$ inside the cylinder, and hence the energy flux into a section of the cylinder of length h and the rate of increase of energy density inside the cylinder. Compare this with the rate of increase of magnetic field energy inside length h of the cylinder.

Solution
(a) From earlier work on electromagnetism we should know that

$$\mathbf{B}(\rho,\phi,z) = \begin{cases} \mu_0 nI\,\widehat{\mathbf{z}} & (\rho < a) \\ 0 & (\rho > a) \end{cases}, \qquad \mathbf{H}(\rho,\phi,z) = \begin{cases} nI\,\widehat{\mathbf{z}} & (\rho < a) \\ 0 & (\rho > a) \end{cases}. \tag{1.52}$$

The magnetic energy density inside the solenoid is

$$u = \frac{B^2}{2\mu_0} = \frac{(\mu_0 nI)^2}{2\mu_0} = \frac{\mu_0 (nI)^2}{2}. \tag{1.53}$$

(b) As the current increases, so does the magnetic field which is uniform and remains in the $\widehat{\mathbf{z}}$ direction. Thus, the induced electric field vector must be perpendicular to the z direction, and from symmetry arguments must be in the $\pm\widehat{\boldsymbol{\phi}}$ direction, i.e. $\mathbf{E}(\mathbf{r},t) = E_\phi(\rho,t)\widehat{\boldsymbol{\phi}}$. Hence, using Faraday's law

$$\oint \mathbf{E}\cdot d\mathbf{r} = -\frac{d}{dt}\Phi_B, \tag{1.54}$$

$$\therefore\quad 2\pi\rho E_\phi(\rho,t) = \begin{cases} -(\pi\rho^2)\mu_0 n\, dI/dt. & (\rho < a) \\ -(\pi a^2)\mu_0 n\, dI/dt. & (\rho > a) \end{cases} \tag{1.55}$$

$$\therefore\quad \mathbf{E}(\mathbf{r},t) = \begin{cases} -(\rho\mu_0 n/2)\, dI/dt\,\widehat{\boldsymbol{\phi}}. & (\rho < a) \\ -(a^2\mu_0 n/2\rho)\, dI/dt\,\widehat{\boldsymbol{\phi}}. & (\rho > a) \end{cases} \tag{1.56}$$

$$\tag{1.57}$$

(c) To find the vector potential we can use

$$\oint_\Gamma \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r} = \int_S \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}.$$

From symmetry arguments, the vector potential must be in $\widehat{\boldsymbol{\phi}}$ direction, and therefore we should use a concentric loop Γ of radius ρ giving

$$2\pi\rho A_\phi = \begin{cases} \pi\rho^2\mu_0 nI & (\rho < a) \\ \pi a^2\mu_0 nI & (\rho > a) \end{cases}, \tag{1.58}$$

$$\therefore\ \mathbf{A} = \begin{cases} (\mu_0 nI\rho/2)\,\widehat{\boldsymbol{\phi}} & (\rho < a) \\ (\mu_0 nIa^2/2\rho)\,\widehat{\boldsymbol{\phi}} & (\rho > a) \end{cases}. \tag{1.59}$$

(d) The Poynting vector at $\rho = a$ is

$$\mathbf{S}(a,\phi,z) = \mathbf{E}(a,\phi,z)\times\mathbf{H}(a,\phi,z), \tag{1.60}$$

$$= \left(-\frac{a}{2}\mu_0 n\frac{dI}{dt}\widehat{\boldsymbol{\phi}}\right)\times\left(nI\,\widehat{\mathbf{z}}\right), \tag{1.61}$$

$$= -\frac{a}{2}\mu_0 n^2 I\frac{dI}{dt}\,\widehat{\boldsymbol{\rho}}, \tag{1.62}$$

$$= -\frac{a}{4}\mu_0\frac{d}{dt}(nI)^2\,\widehat{\boldsymbol{\rho}}, \tag{1.63}$$

and is directed radially inwards.

To find the energy flux going into a section of the solenoid of length h we multiply by the its surface area $2\pi ah$

$$2\pi ah\left[-S_\rho(a,\phi,z)\right] \;=\; 2\pi ah\frac{a}{4}\frac{d}{dt}\left[\mu_0(nI)^2\right] \;=\; \pi a^2 h\frac{d}{dt}\left[\frac{\mu_0 H^2}{2}\right] \tag{1.64}$$

which is just the rate of increase of magnetic energy inside length h of the solenoid.

1–9 Using the Maxwell stress tensor find the pressure exerted on a perfectly absorbing screen by an electromagnetic plane wave at normal incidence.

Solution

Assume the screen is in the x–y plane and the EM wave is

$$\mathbf{E} = E_0\widehat{\mathbf{x}}\cos(kz-\omega t), \qquad \mathbf{B} = \frac{E_0}{c}\widehat{\mathbf{y}}\cos(kz-\omega t). \tag{1.65}$$

Momentum conservation in electrodynamics is expressed by

$$\frac{d}{dt}\left[\mathbf{P}_{\text{part}} + \int_V (\varepsilon_0\mathbf{E}\times\mathbf{B})\,d^3r\right] = \sum_i \widehat{\mathbf{e}}_i \oint_S \left(\sum_j T_{ij}\widehat{\mathbf{e}}_j\cdot\widehat{\mathbf{n}}\right)dS, \tag{1.66}$$

where $\widehat{\mathbf{n}}$ is the outward normal unit vector at surface S which bounds volume V, and

$$T_{ij} = \varepsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}\delta_{ij}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\right]. \tag{1.67}$$

The volume V corresponds to the space defined by $z>0$, so $\widehat{\mathbf{n}} = -\widehat{\mathbf{z}}$ and $\widehat{\mathbf{e}}_j\cdot\widehat{\mathbf{n}} \neq 0$ for $j \leftrightarrow z$. Hence we need only calculate T_{xz}, T_{yz} and T_{zz}.

For the EM wave above $T_{xz} = T_{yz} = 0$ and

$$T_{zz} \;=\; -\frac{1}{2}\left(\varepsilon_0\mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \;=\; -u \tag{1.68}$$

where u is the energy density of the EM wave.

The pressure is given by the rate of change of the momentum of particles and fields inside V, so we need to integrate over a portion of the surface, i.e. some area A of the x—y plane, and with V bounded by A and extending in the $+z$ direction. Then the force on the contents of V is

$$\mathbf{F} = \sum_i \widehat{\mathbf{e}}_i \int_A \left(\sum_j T_{ij} \widehat{\mathbf{e}}_j \cdot \widehat{\mathbf{n}} \right) dS, \tag{1.69}$$

$$= \widehat{\mathbf{z}} \left[T_{zz} \widehat{\mathbf{z}} \cdot (-\widehat{\mathbf{z}}) \right] A \tag{1.70}$$

$$= -\widehat{\mathbf{z}} \, (-u) \, A \tag{1.71}$$

$$= uA \, \widehat{\mathbf{z}}. \tag{1.72}$$

Hence, the pressure $p = F/A = u$ is equal to the energy density in the EM wave.

If we had not been asked explicitly to use the Maxwell stress tensor we could have arrived at the same result in a simpler way by noting that the momentum density is $\mathbf{g} = \mathbf{S}/c^2$, and that for an EM wave $\mathbf{S} = u\, c\, \widehat{\mathbf{k}}$. Pressure will be the magnitude of the momentum flux $gc = (S/c^2)c = u$.

2 Electromagnetic waves in empty space and linear dielectrics

2–1 Prove that the spherical waves given by

$$f(\mathbf{r},t) = f_0\, r^{-1} \exp[i(\pm kr - \omega t)] \tag{2.1}$$

are solutions of the 3D wave equation.

Solution

We need to show that this is a solution of the three-dimensional wave equation

$$\nabla^2 f \;=\; \frac{1}{v_p^2}\frac{\partial^2 f}{\partial t^2}. \tag{2.2}$$

Using the Laplacian in spherical coordinates and noting that f has no dependence on θ or ϕ then the left hand side of Eq. 2.2 is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right), \tag{2.3}$$

$$= f_0\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(-r^{-2} \pm r^{-1} ik\right)\exp[i(\pm kr - \omega t)]\right\}, \tag{2.4}$$

$$= f_0\frac{1}{r^2}\frac{\partial}{\partial r}\left\{(-1 \pm rik)\exp[i(\pm kr - \omega t)]\right\}, \tag{2.5}$$

$$= f_0\frac{1}{r^2}\left\{\left(\mp ik \pm ik - rk^2\right)\exp[i(\pm kr - \omega t)]\right\}, \tag{2.6}$$

$$= -k^2 f. \tag{2.7}$$

Now since $v_p = \omega/k$ the right hand side of Eq. 2.2 is

$$\frac{1}{v_p^2}\frac{\partial^2 f}{\partial t^2} \;=\; -\frac{1}{v_p^2}\omega^2 f \;=\; -k^2 f \tag{2.8}$$

and so the 3D wave equation is satisfied.

2–2 A monochromatic plane wave $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ is travelling in the $+\widehat{\mathbf{z}}$ direction through a lossless linear medium with relative permittivity $\varepsilon_r = 4$ and relative permeability $\mu_r \approx 1$ and is polarised in the $\widehat{\mathbf{x}}$ direction. The frequency is $\nu = 1$ GHz and E has a

maximum value of $+10^{-3}$ V m^{-1} at $t = 5$ ns and $z = 1$ m.
(a) Find the angular frequency, phase velocity, wavenumber, wave vector, and wavelength.
(b) Obtain the instantaneous expression for $\mathbf{E}(\mathbf{r}, t)$ valid for any position and time.
(c) Obtain the instantaneous expression for $\mathbf{H}(\mathbf{r}, t)$ valid for any position and time.
(d) Find the Poynting vector and its time-averaged value.
(e) Find the locations where E_x is maximum when $t = 0$ s.

Solution

(a) The angular frequency is $\omega = 2\pi\nu = 2\pi \times 10^9 = 6.24 \times 10^9$ rad/s.

The phase velocity is

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{1 \times 4}} = 1.5 \times 10^8 \text{ m/s}. \tag{2.9}$$

The wavenumber is

$$k = \frac{\omega}{v_p} = \frac{6.238 \times 10^9}{1.5 \times 10^8} = 41.9 \ \text{ rad/m}. \tag{2.10}$$

The wave vector has magnitude k and is in the direction of wave propagation, i.e. $\mathbf{k} = 41.89\,\widehat{\mathbf{z}}$ rad/m.

The wavelength is $\lambda = 2\pi/k = v_p/\nu = 1.5 \times 10^8/10^9 = 0.15$ m.

(b) Taking the real part of the monochromatic plane wave, and noting that the wave is travelling in the $+\widehat{\mathbf{z}}$ direction, has maximum $E_0 = 10^{-3}$ V m^{-1} and is polarised in the $\widehat{\mathbf{x}}$ direction,

$$\mathbf{E}(\mathbf{r}, t) = 10^{-3} \cos\left[41.89z - 6.238 \times 10^9 t + \delta\right] \widehat{\mathbf{x}} \quad \text{V m}^{-1}. \tag{2.11}$$

To find the phase constant δ we use the information that E is maximum when $t = 5$ ns and $z = 1$ m, and this occurs when the phase is $2n\pi$ (n is any integer), so

$$41.89 \times 1 - 6.238 \times 10^9 \times 5 \times 10^{-9} + \delta = 2n\pi \tag{2.12}$$

giving $\delta = (2n\pi - 10.70)$ rad. The fields are unaffected by our choice of integer n. So, for convenience, setting $n = 0$ we find

$$\mathbf{E}(\mathbf{r}, t) = 10^{-3} \cos\left[41.89z - 6.238 \times 10^9 t - 10.70\right] \widehat{\mathbf{x}} \quad \text{V/m}. \tag{2.13}$$

(c) We use $\mathbf{B} = \mathbf{k} \times \mathbf{E}/\omega = \widehat{\mathbf{k}} \times \mathbf{E}/v_p$, then

$$\mathbf{B}(\mathbf{r},t) = \widehat{\mathbf{z}} \times \left[10^{-3} \cos\left(41.89z - 6.238 \times 10^9 t - 10.70\right) \widehat{\mathbf{x}}\right] / 1.5 \times 10^8 \quad \mathrm{T}, \tag{2.14}$$

$$= 6.67 \times 10^{-12} \cos\left(41.89z - 6.238 \times 10^9 t - 10.70\right) \widehat{\mathbf{y}} \quad \mathrm{T}. \tag{2.15}$$

Now we can readily obtain $\mathbf{H} = \mathbf{B}/\mu = \mathbf{B}/\mu_0$,

$$\mathbf{H}(\mathbf{r},t) = 5.31 \times 10^{-6} \cos\left(41.89z - 6.238 \times 10^9 t - 10.70\right) \widehat{\mathbf{y}} \quad \mathrm{A/m}. \tag{2.16}$$

(d) The Poynting vector is

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E} \times \mathbf{H}, \tag{2.17}$$

$$= \left(10^{-3}\,\widehat{\mathbf{x}}\right) \times \left(5.31 \times 10^{-6} \widehat{\mathbf{y}}\right) \cos^2\left(41.89z - 6.238 \times 10^9 t - 10.70\right), \tag{2.18}$$

$$= 5.31 \times 10^{-9} \cos^2\left(41.89z - 6.238 \times 10^9 t - 10.70\right) \widehat{\mathbf{z}} \quad \mathrm{W/m^2}. \tag{2.19}$$

Because the time-averaged value of $\cos^2(\omega t)$ is $1/2$, we have

$$\langle \mathbf{S} \rangle = 2.65 \times 10^{-9}\,\widehat{\mathbf{z}} \;\; \mathrm{W/m^2}. \tag{2.20}$$

(e) We need to find where E_x is maximum at $t = 0$. This occurs where the phase is $2n\pi$ with n being an integer, i.e. E_x is maximum where

$$2n\pi = 41.89 \times z - (6.238 \times 10^9) \times 0 - 10.70. \tag{2.21}$$

$$\therefore \;\; z \;=\; (10.70 + 2n\pi)/41.89 \;=\; 0.255 + 0.15\,n \quad \mathrm{m}. \tag{2.22}$$

Note that the term $0.15\,n$ is simply $n\lambda$.

2–3 Given the electric fields for the following polarisations,

$$\mathbf{E}(\mathbf{r}, t) = (\widehat{\mathbf{x}} E_{0,x} + \widehat{\mathbf{y}} E_{0,y}) \cos(kz - \omega t + \delta) \quad \text{(linear)}, \tag{2.23}$$

$$\mathbf{E}(\mathbf{r}, t) = E_0[\widehat{\mathbf{x}} \cos(kz - \omega t + \delta_x) - \widehat{\mathbf{y}} \sin(kz - \omega t + \delta_x)] \quad \text{(left circular)}, \tag{2.24}$$

$$\mathbf{E}(\mathbf{r}, t) = \widehat{\mathbf{x}} E_{0,x} \cos(kz - \omega t + \delta_x) + \widehat{\mathbf{y}} E_{0,y} \sin(kz - \omega t + \delta_x) \quad \text{(right elliptical)} \tag{2.25}$$

with $E_{0,y} > E_{0,x}$, find the instantaneous and time-averaged energy densities and Poynting vectors. In each case, assume the wave is propagating in a medium with permittivity ε and permeability μ.

<u>Solution</u>

The energy density, magnetic field, magnetic intensity and Poynting vector of an EM wave are given by

$$u(\mathbf{r}, t) \;=\; \varepsilon |\mathbf{E}(\mathbf{r}, t)|^2, \tag{2.26}$$

$$\mathbf{B}(\mathbf{r}, t) \;=\; \omega^{-1} \mathbf{k} \times \mathbf{E}(\mathbf{r}, t) \;=\; \frac{1}{v_p} \widehat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \;=\; \sqrt{\varepsilon\mu}\, \widehat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t), \tag{2.27}$$

$$\mathbf{H}(\mathbf{r}, t) \;=\; \frac{\mathbf{B}(\mathbf{r}, t)}{\mu} \;=\; \sqrt{\frac{\varepsilon}{\mu}} \widehat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \;=\; Z \widehat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t), \tag{2.28}$$

$$\mathbf{S}(\mathbf{r}, t) \;=\; \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \;=\; [\mathbf{E}(\mathbf{r}, t)] \times \left[Z \widehat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \right] \;=\; Z |\mathbf{E}(\mathbf{r}, t)|^2 \widehat{\mathbf{k}}, \tag{2.29}$$

where $v_p = 1/\sqrt{\mu\varepsilon}$ is the phase velocity and $Z = \sqrt{\varepsilon/\mu}$ is the wave impedance of the medium.

(a) Linear polarisation:

$$\mathbf{E}(\mathbf{r},t) = (\widehat{\mathbf{x}}E_{0,x} + \widehat{\mathbf{y}}E_{0,y})\cos(kz - \omega t + \delta), \tag{2.30}$$

$$u(\mathbf{r},t) = \varepsilon(E_{0,x}^2 + E_{0,y}^2)\cos^2(kz - \omega t + \delta), \tag{2.31}$$

$$\langle u\rangle = \frac{1}{2}\varepsilon(E_{0,x}^2 + E_{0,y}^2), \tag{2.32}$$

$$\mathbf{S}(\mathbf{r},t) = Z(E_{0,x}^2 + E_{0,y}^2)\cos^2(kz - \omega t + \delta)\,\widehat{\mathbf{z}}, \tag{2.33}$$

$$\langle S\rangle = \frac{1}{2}Z(E_{0,x}^2 + E_{0,y}^2)\,\widehat{\mathbf{z}}. \tag{2.34}$$

(b) Left circular polarisation:

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= E_0[\widehat{\mathbf{x}}\cos(kz - \omega t + \delta_x) - \widehat{\mathbf{y}}\sin(kz - \omega t + \delta_x)],\\
u(\mathbf{r},t) &= \varepsilon E_0^2[\cos^2(kz - \omega t + \delta_x) + \sin^2(kz - \omega t + \delta_x)] \quad = \quad \varepsilon E_0^2,\\
\langle u\rangle &= \varepsilon E_0^2,\\
\mathbf{S}(\mathbf{r},t) &= ZE_0^2\widehat{\mathbf{z}},\\
\langle \mathbf{S}\rangle &= ZE_0^2\,\widehat{\mathbf{z}}.
\end{aligned}$$

(b) Right elliptical polarisation ($E_{0,y} > E_{0,x}$):

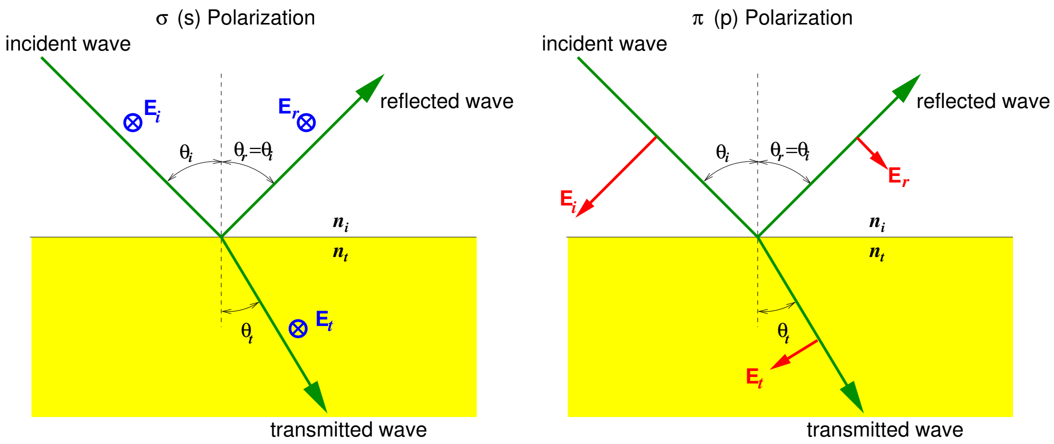
$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \widehat{\mathbf{x}}E_{0,x}\cos(kz - \omega t + \delta_x) \quad + \quad \widehat{\mathbf{y}}E_{0,y}\sin(kz - \omega t + \delta_x),\\
u(\mathbf{r},t) &= \varepsilon[E_{0,x}^2\cos^2(kz - \omega t + \delta_x) \quad + \quad E_{0,y}^2\sin^2(kz - \omega t + \delta_x)],\\
\therefore\quad u(\mathbf{r},t) &= \varepsilon\left[E_{0,x}^2 + (E_{0,y}^2 - E_{0,x}^2)\sin^2(kz - \omega t + \delta_x)\right],\\
\langle u\rangle &= \frac{1}{2}\varepsilon(E_{0,x}^2 + E_{0,y}^2),\\
\mathbf{S}(\mathbf{r},t) &= Z\left[E_{0,x}^2 \quad + \quad (E_{0,y}^2 - E_{0,x}^2)\sin^2(kz - \omega t + \delta_x)\right]\,\widehat{\mathbf{z}},\\
\langle \mathbf{S}\rangle &= \frac{1}{2}\,Z\,(E_{0,x}^2 + E_{0,y}^2)\widehat{\mathbf{z}}.
\end{aligned}$$

2–4 For a monochromatic EM plane wave incident on a plane interface between two dielectrics describe what is meant by perpendicular (σ, s) and parallel (π, p) polarisation states.

Include a suitable diagram in your answer.

Solution

In transmission/reflection of light at an interface between two dielectrics, the incident ray, transmitted ray and reflected ray are in the plane of incidence, which is defined as the plane containing the incident ray and the normal to the interface. If the plane in which the electric field oscillates is perpendicular to the plane of incidence, the wave has perpendicular (σ, s) polarisation, and if the the plane in which the electric field oscillates is parallel to the plane of incidence, the wave has parallel (π, p) polarisation.



2–5 Define what is meant by the amplitude reflection and transmission coefficients, and by the reflectance and transmittance.

Solution

The amplitude reflection and transmission coefficients relate the complex amplitudes of the electric fields of the reflected and transmitted waves to the complex amplitude of the electric field of the incident wave. For example for perpendicular polarisation,

$$E_r^\perp = r_\perp E_i^\perp, \quad E_t^\perp = t_\perp E_i^\perp.$$

Reflectance is defined by the ratio of the the energy flux reflected by unit area of the interface to the energy flux incident on unit area of the interface.

Transmittance is defined by the ratio of the the energy flux transmitted through unit area of the interface to the energy flux incident on unit area of the interface.

2–6 (a) Use the amplitude reflection coefficient for parallel polarisation

$$r_\parallel(\theta_i) \equiv \frac{E_r^\parallel}{E_i^\parallel} = \frac{(n_t/n_i)^2\cos\theta_i - \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}{(n_t/n_i)^2\cos\theta_i + \sqrt{(n_t/n_i)^2 - \sin^2\theta_i}}. \tag{2.35}$$

to show that Brewster's angle is given by both

$$\cos\theta_B = \frac{1}{\sqrt{1 + (n_t/n_i)^2}} \quad \text{and} \quad \tan\theta_B = \frac{n_t}{n_i}. \tag{2.36}$$

(b) Show that

$$\tan\theta_B = \frac{n_t}{n_i} \tag{2.37}$$

can also be derived from Snell's law and by requiring the angle between the reflected and transmitted rays to be $\pi/2$.

Solution

(a) Brewster's angle is the angle of incidence such that $r_{\|}(\theta_B) = 0$, so that

$$(n_t/n_i)^2 \cos\theta_B = \sqrt{(n_t/n_i)^2 - \sin^2\theta_B}, \tag{2.38}$$

$$(n_t/n_i)^4 \cos^2\theta_B = (n_t/n_i)^2 - (1 - \cos^2\theta_B), \tag{2.39}$$

$$\cos^2\theta_B \left[(n_t/n_i)^4 - 1\right] = (n_t/n_i)^2 - 1, \tag{2.40}$$

$$\cos^2\theta_B \left[(n_t/n_i)^2 - 1\right]\left[(n_t/n_i)^2 + 1\right] = (n_t/n_i)^2 - 1, \tag{2.41}$$

$$\cos^2\theta_B \left[(n_t/n_i)^2 + 1\right] = 1, \tag{2.42}$$

$$\therefore \quad \cos\theta_B = \frac{1}{\sqrt{1 + (n_t/n_i)^2}}. \tag{2.43}$$

The alternative formula we get from Eq. 2.42 above, from which

$$\cos^2\theta_B\,(n_t/n_i)^2 = 1 - \cos^2\theta_B, \tag{2.44}$$

$$\cos^2\theta_B\,(n_t/n_i)^2 = \sin^2\theta_B, \tag{2.45}$$

$$\therefore \quad \tan\theta_B = \frac{n_t}{n_i}. \tag{2.46}$$

(b) Eq. 2.46 can be derived from Snell's law and requiring the angle between the reflected and transmitted rays to be $\pi/2$. For this case $\theta_t = (\pi/2 - \theta_i)$, and so from Snell's law we find

$$n_i \sin\theta_i = n_t \sin\theta_t, \tag{2.47}$$

$$n_i \sin\theta_B = n_t \sin\left(\frac{\pi}{2} - \theta_B\right), \tag{2.48}$$

$$n_i \sin\theta_B = n_t \cos\theta_B, \tag{2.49}$$

$$\therefore \quad \tan\theta_B = \frac{n_t}{n_i}. \tag{2.50}$$

2–7 Give a formula with explanations for the reflectance and transmittance in terms of the magnitude of the Poynting vectors of the various waves, and their angles with respect to the normal to the interface. How is the reflectance related to the amplitude reflection coefficient? How is the transmittance related to the reflectance? Why?

Solution

In terms of the Poynting vectors of the three waves,

$$R = \frac{S_r \cos\theta_r}{S_i \cos\theta_i}, \quad T = \frac{S_t \cos\theta_t}{S_i \cos\theta_i}.$$

Note that the area of the interface as seen by each of the three waves is the projected area, and the hence the relevant $\cos\theta$ factor is needed.

The reflectance is related to the amplitude reflection coefficient by $R = |r|^2$.

The transmittance is related to the reflectance by $R + T = 1$ as required by energy conservation.

2–8 Explain the physical meaning of the critical angle, and derive its formula.

Solution

From Snell's law $n_t \sin\theta_t = n_i \sin\theta_i$

$$\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i.$$

For internal reflection, Snell's law can not be satisfied for angles of incidence greater than the critical angle,

$$\theta_c = \arcsin\left(\frac{n_t}{n_i}\right),$$

for which $\sin\theta_t \geq 1$. For larger angles of incidence, there will be no transmitted wave, i.e., there will be total internal reflection.

3 Electromagnetic waves in dispersive media

3–1 The resistivity of silver is $\rho = 1.6 \times 10^{-8}\,\Omega$ m, and its permeability is $\mu = 0.9998\mu_0$, find the reflectance for light of wavelength 500 nm at normal incidence from air.

Solution

For perpendicular and parallel polarisations the amplitude reflection coefficients for normal incidence are

$$r_\perp(0°) \;=\; \frac{1 - n_t/n_i}{1 + n_t/n_i}, \qquad r_\parallel(0°) \;=\; \frac{n_t/n_i - 1}{n_t/n_i + 1}, \tag{3.1}$$

and we see that $|r_\perp(0°)| = |r_\parallel(0°)|$. Hence, we can use either amplitude reflection coefficient in calculating the reflectance, as we must be able to do since parallel and perpendicular polarisations are undefined for $\theta_i = 0°$. Then, for external reflection from air we have

$$r_\perp(0°) \;=\; \frac{1 - n(\omega)}{1 + n(\omega)} \;=\; \frac{1 - \mathrm{Re}\{n\} - i\,\mathrm{Im}\{n\}}{1 + \mathrm{Re}\{n\} + i\,\mathrm{Im}\{n\}}. \tag{3.2}$$

Using the rule that the complex conjugate of a quotient of two complex numbers is the quotient of the complex conjugates of the two complex numbers, the reflectance is

$$R(0°) = |r(0°)|^2 = \frac{(1 - \mathrm{Re}\{n\})^2 + (\mathrm{Im}\{n\})^2}{(1 + \mathrm{Re}\{n\})^2 + (\mathrm{Im}\{n\})^2}. \tag{3.3}$$

From the wave number of a good conductor

$$k(\omega) \approx (1 + i)2^{-1/2}(\mu\sigma\omega)^{1/2} \tag{3.4}$$

we are able to estimate the refractive index,

$$n(\omega) \approx (1 + i)c\sqrt{\frac{\mu\sigma}{2\omega}}. \tag{3.5}$$

A wavelength of 500 nm corresponds to an angular frequency of $\omega = 2\pi \times 3 \times 10^8/500 \times 10^{-9} = 3.77 \times 10^{15}$ rad s^{-1}. We see that for silver

$$\mathrm{Re}\{n\} \approx \mathrm{Im}\{n\} = 3 \times 10^8 \sqrt{\frac{0.9998 \times (4\pi \times 10^{-7}) \times (1/1.6 \times 10^{-8})}{2 \times (3.77 \times 10^{15})}} = 30.6. \tag{3.6}$$

Hence, using Eq. 3.3 the reflectance is

$$R(0^\circ) \;=\; \frac{(1-\mathrm{Re}\{n\})^2+(\mathrm{Im}\{n\})^2}{(1+\mathrm{Re}\{n\})^2+(\mathrm{Im}\{n\})^2} \;=\; 0.937\;. \tag{3.7}$$

3–2 Show that the time-averaged power density $\langle \mathbf{E}\cdot\mathbf{J}\rangle$ in a dilute plasma is zero.

Solution

The conductivity of a dilute plasma is purely imaginary,

$$\sigma = i\frac{n_e e^2}{m_e\omega}, \tag{3.8}$$

so that the current density and electric field are 90° out of phase. Consequently, the time-averaged power will be zero, as shown below

$$\mathbf{E}\cdot\mathbf{J} = \mathrm{Re}\{\mathbf{E}_0 e^{-i\omega t}\}\cdot \mathrm{Re}\left\{e^{i\pi/2}\,|\sigma|\,\mathbf{E}_0 e^{-i\omega t}\right\}, \tag{3.9}$$

$$= \{\mathbf{E}_0\cos(-\omega t)\}\cdot\left\{|\sigma|\,\mathbf{E}_0\cos\left(\frac{\pi}{2}-\omega t\right)\right\}, \tag{3.10}$$

$$= |\sigma|\,E_0^2\cos(\omega t)\sin(\omega t)\,, \tag{3.11}$$

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{2}\,|\sigma|\,\mathbf{E}_0^2\sin(2\omega t) \tag{3.12}$$

$$\langle\mathbf{E}\cdot\mathbf{J}\rangle = 0. \tag{3.13}$$

Hence, there is no resistive energy loss.

3–3 At midnight and noon, at a certain location and date, the electron number density in the ionosphere was as given in the plot below.

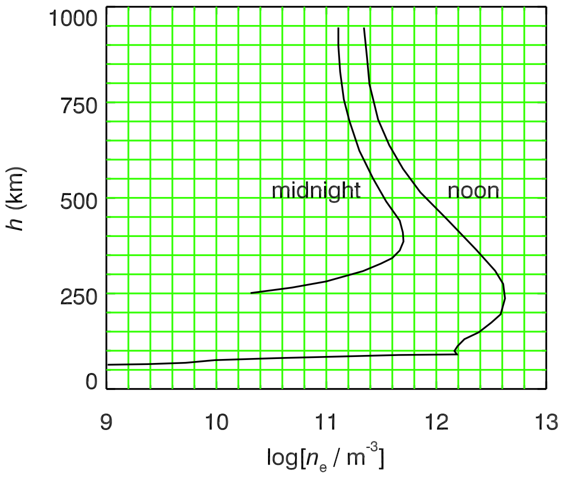
(a) Label the x-axis at the top of the plot in terms of the plasma frequency ν_p (MHz).

(b) Find the height of the reflecting layer at 1 MHz and 3 MHz at midnight and noon.

(c) Find the minimum frequencies for communication with orbiting satellites at midnight and noon.

(d) Estimate the total electron content in (electrons m^{-2}) at midnight and noon.

(e) An instantaneous broad-band pulse of radio-frequency interference (RFI) is emitted overhead by an orbiting satellite and observed by an terrestrial detector with a bandwidth from 1.2—1.8 GHz. What is the duration of the observed pulse?

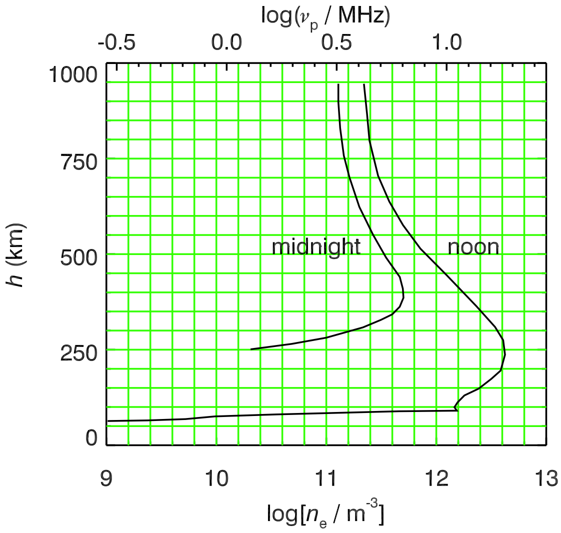


Solution

(a) The plasma frequency in Hz is $\nu_p = \omega_p / 2\pi$

$$\nu_p = \frac{1}{2\pi}\left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2} \tag{3.14}$$

and so for $n_e = 10^9$ m^{-3} the plasma frequency is 2.8×10^5 Hz and $\log(\nu_p/\mathrm{MHz}) = -0.55$. Because $\nu_p \propto n_e^{1/2}$ two decades of n_e correspond to one decade of ν_p, so we now have the information needed to relabel the plot as shown below.



(b) Reading off the plot, we obtain the height of the reflecting layer shown in the table below.

Time	1 MHz	3 MHz
midnight	240 km	300 km
noon	80 km	85 km

(c) Reading off the plasma frequencies at the maximum n_e values, the minimum frequencies to avoid reflection by the ionosphere are 6.3 MHz (midnight) and 18 MHz (noon).

(d) The total electron content (TEC) is $\int n_e(h)dh$, which we can estimate by tabulating n_e (in units of 10^{12} m^{-3}) at every 100 km, and multiplying the sum by $\Delta h = 10^5$ m as follows.

h (km)	noon	midnight
100	1.4	
200	3.9	
300	3.6	0.17
400	1.8	0.50
500	0.8	0.34
600	0.4	0.22
700	0.3	0.16
800	0.3	0.14
900	0.2	0.13
Sum	12.7	1.66

Hence, the TEC at noon is $(12.7 \times 10^{12}) \times 10^5 = 1.27 \times 10^{18}$ electrons m^{-2}, whereas at midnight the TEC is 1.66×10^{17} electrons m^{-2}.

(e) Using the result for a given dispersion measure, which is defined in a similar way to the TEC, we have the arrival time of the pulse at a given frequency

$$t(\nu) = \frac{e^2}{8\pi^2 \varepsilon_0 m_e c} \times \text{TEC} \times \nu^{-2} = (1.34 \times 10^{-7}) \times \text{TEC}\ \nu^{-2} \tag{3.15}$$

such that the spread in arrival times is

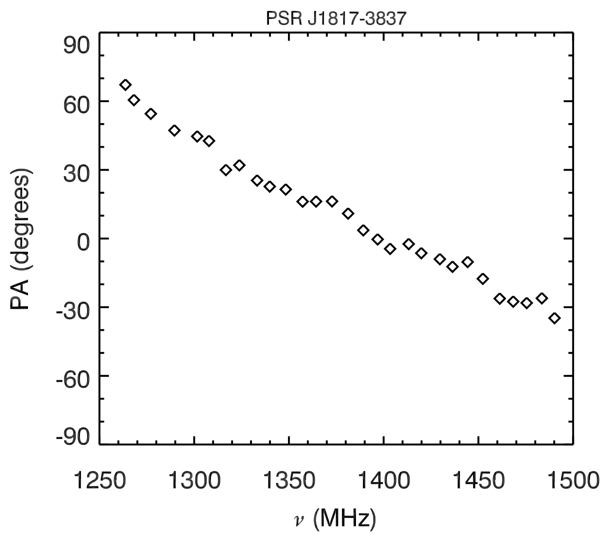
$$\Delta t = (1.34 \times 10^{-7})(\nu_1^{-2} - \nu_2^{-2}) \times \text{TEC} \tag{3.16}$$

$$= (1.34 \times 10^{-7}) \times [(1.2 \times 10^9)^{-2} - (1.8 \times 10^9)^{-2}] \times \text{TEC} \tag{3.17}$$

$$= 2.46 \times 10^{-26} \times \text{TEC}. \tag{3.18}$$

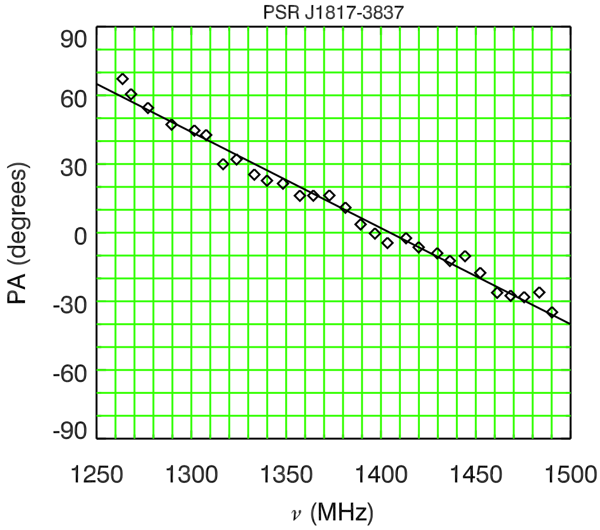
Hence, the pulse is spread out in time by 3.13×10^{-8} s (noon) or 4.09×10^{-9} s (midnight).

3–4 For pulsar PSR J1817-3837 the observed position angle (PA) of linear polarisation with respect to the North Celestial Pole is plotted below vs. frequency. The dispersion measure is 102.85 pc cm^{-3}, where 1 pc=3.09×10^{16} m (1 parsec) is the distance unit used by astronomers. Find the average value of the parallel component of the magnetic field along the line of sight to this pulsar.



Solution

The first step is to determine the rotation measure for this pulsar using the data given in the plot. Even though we expect a ν^{-2} dependence of position angle, the range of frequency is sufficiently small that we can make a linear fit (shown below).



Reading off data at two frequencies we can tabulate it together with wavelength as shown.

ν (MHz)	λ (m)	λ^2 (m^2)	PA (degrees)
1500	0.2	0.04	-40
1250	0.24	0.0576	+64

The observed rotation measure is

$$RM = \frac{\Delta\,\mathrm{PA}}{\Delta\,\lambda^2} = \frac{104^\circ}{0.0176\ \mathrm{m}^2} = 5909^\circ\,\mathrm{m}^{-2} = 103\,\mathrm{rad}\,\mathrm{m}^{-2}. \tag{3.19}$$

The rotation measure is related to the magnetic field and electron density along the line of sight to the pulsar by

$$RM = 2.63\times10^{-13}\int n_e(\mathbf{r})\,\mathbf{B}(\mathbf{r})\cdot d\mathbf{r} \quad (\mathrm{rad\ m}^{-2}). \tag{3.20}$$

Hence, for PSR J1817-3837 we must have

$$\int n_e(\mathbf{r})\,\mathbf{B}(\mathbf{r})\cdot d\mathbf{r} = RM/(2.63\times10^{-13}) = 3.92\times10^{14} \quad \mathrm{T\ m}^{-2}. \tag{3.21}$$

The observed dispersion measure is

$$DM = \int n_e(\mathbf{r})\,dr = 102.85\times(3.09\times10^{16})\times10^6 = 3.18\times10^{24} \quad \mathrm{m}^{-2}. \tag{3.22}$$

Hence, the average value of the parallel component of the magnetic field along the line of sight to this pulsar is

$$\langle \mathbf{B} \cdot \widehat{\mathbf{r}} \rangle = \frac{\int n_e(\mathbf{r})\, \mathbf{B}(\mathbf{r}) \cdot d\mathbf{r}}{\int n_e(\mathbf{r})\, dr} \quad = \quad 1.23 \times 10^{-10} \quad \mathrm{T}. \tag{3.23}$$

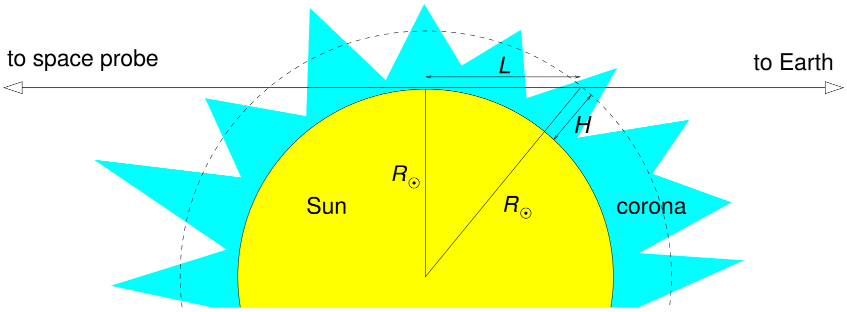
3–5 We are receiving radio signals from an interplanetary space probe which is far from the Sun. It is currently on a trajectory such that it has been eclipsed by the Sun, but it is now emerging from behind the Sun. The space probe regularly broadcasts instantaneous broadband pulses of radio-frequency emission every second, and it also broadcasts spacecraft experiment instrument data over a narrow frequency band of width 100 kHz extending from 10,000.0 MHz to 10,000.1 MHz. It transmits using a dipole antenna aligned perpendicular to the plane of the ecliptic (the plane containing the Earth's orbit). We are just renewing radio contact as the space probe is emerging from behind the solar limb.

The Sun's corona is an extremely hot ($> 10^6$ K) plasma which is highly variable, and has dynamic coronal loops of magnetised plasma. The corona is located above the Sun's photosphere. At the time we are observing, the base of the corona in front of the space probe as it emerges from behind the Sun has an electron number density of $n_e(R_\odot) \sim 3\times 10^{15}$ m^{-3}. The corona's a scale height is about $H \sim 10^8$ m, such that the electron number density decreases with height as $n_e(r) = n_e(R_\odot)\exp[-(r - R_\odot)/H]$. Imagine that, at the time we are observing, the Sun's magnetic dipole moment is $m = 2 \times 10^{29}$ A m^2 and it happens to be pointing directly towards Earth.

(a) What is the minimum radio frequency that we are able to receive from the space probe?
(b) What is the additional delay of the pulses due to propagation through the corona?
(c) By how much has the plane of polarisation of the EM wave carrying the data signal rotated due to propagation through the corona?
[Make what you think are reasonable approximations – do not attempt a rigorous calculation. The radius of the solar photosphere is $R_\odot = 6.955 \times 10^8$ m.]

Solution

The geometry is as shown below.



(a) The plasma frequency is

$$\nu_p = \frac{1}{2\pi}\left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2} \tag{3.24}$$

and so at the base of the corona where $n_e(R_\odot) = 3 \times 10^{15}$ m^{-3} the plasma frequency is 3.1 GHz. No communication can take place below this frequency.

(b) To estimate the dispersion measure we could use the maximum electron number density and a path-length of $2L$ corresponding to the path through the corona below height H (see diagram above), where

$$L = \sqrt{(R_\odot + H)^2 - R_\odot^2} = 3.9 \times 10^8 \ \text{m}. \tag{3.25}$$

Hence, the dispersion measure is $\text{DM} \sim 2\,L\,n_e(R_\odot) \sim 2.3 \times 10^{24} \ \text{m}^{-2}$.

Using the standard result for the time delay for a given dispersion measure, we have the arrival time of the pulse at different frequencies

$$t(\nu) \ = \ \frac{e^2}{8\pi^2\varepsilon_0 m_e c} \times \text{DM} \times \nu^{-2} \ = \ (1.34 \times 10^{-7}) \times \text{DM}\,\nu^{-2} \tag{3.26}$$

Over the bandwidth of the detector the delays are

$$t(10,000.0\,\text{MHz}) = (1.34\times10^{-7})\times(2.3\times10^{24})\times(1.0000\times10^{10})^{-2} = 3.0820\times10^{-3}\ \text{s},$$
$$t(10,000.1\,\text{MHz}) = (1.34\times10^{-7})\times(2.3\times10^{24})\times(1.0001\times10^{10})^{-2} = 3.0814\times10^{-3}\ \text{s}.$$

Hence, the start of the observed pulse is delayed by ~ 3 ms and the duration of the pulse is $\sim 600\ \mu$s.

(c) To find the angle of rotation of the plane of polarisation, we need to estimate the parallel component of magnetic field by, say, the surface magnetic field at the magnetic equator $\mathbf{r} = (R_\odot, 90°, \phi)$, where the pole of these spherical coordinates corresponds to the magnetic axis of the Sun. Then, we can obtain the surface field from the magnetic dipole moment $\mathbf{m}_\odot$ using

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{m}_\odot \cdot \mathbf{r}) - r^2\mathbf{m}_\odot}{r^5}\right], \tag{3.27}$$

and so at the Sun's magnetic equator the surface magnetic field magnitude is

$$B \ = \ \frac{\mu_0}{4\pi}\frac{m_\odot}{R_\odot^3} \ = \ \frac{4\pi \times 10^{-7}}{4\pi}\frac{2 \times 10^{29}}{(6.996 \times 10^8)^3} \ = \ 5.8 \times 10^{-5}\ \text{T}. \tag{3.28}$$

The rotation measure is related to the magnetic field and electron density along the line of sight to the pulsar by

$$
\begin{aligned}
RM &= 2.63 \times 10^{-13} \int n_e(\mathbf{r})\,\mathbf{B}(\mathbf{r}) \cdot d\mathbf{r} \quad (\text{rad m}^{-2}) && (3.29)\\
&= (2.63 \times 10^{-13}) \times (3 \times 10^{15}) \times (5.8 \times 10^{-5}) \times (2\ \times\ 3.9 \times 10^{8}) && (3.30)\\
&= 3.6 \times 10^{7}\ \text{ rad m}^{-2}. && (3.31)
\end{aligned}
$$

The rotation measure is defined through

$$
RM = \frac{\Delta\,\mathrm{PA}}{\Delta\,\lambda^2}. \qquad (3.32)
$$

Hence, the plane of polarisation will have rotated by

$$
\begin{aligned}
\mathrm{PA}(10,000.0\,\mathrm{MHz}) &= (3.6 \times 10^{7}) \times \left(\frac{3 \times 10^{8})}{10^{10}}\right)^2 = 32351.522 \text{ rad}, && (3.33)\\
\mathrm{PA}(10,000.1\,\mathrm{MHz}) &= (3.6 \times 10^{7}) \times \left(\frac{3 \times 10^{8})}{1.0001 \times 10^{10}}\right)^2 = 32345.053 \text{ rad}. && (3.34)
\end{aligned}
$$

So the plane of polarisation will have rotated by an enormous angle of the order of 30,000 radians, and we see that over the 100 kHz receiver bandwidth the position angle changes by $\sim$ 5 radians. These numbers are of course only order of magnitude estimates, but they show that trying to align the a dipole antenna with the plane of polarisation makes no sense, and that to obtain all the signal it would be necessary to have two orthogonal polarisations and appropriately combine the two signals.

4 Waveguides

4–1 A standard waveguide for the "X-band" (8.2–12.4 GHz) has internal cross section 2.286 cm by 1.143 cm. Find the the first two cut-off frequencies, as these will give the frequency range for which only the TE_{10} mode will propagate.

Solution

Since $a = 2b$, these will correspond to $\omega_{10} = c\pi/a$ and $\omega_{01} = \omega_{20} = 2c\pi/a = 2\omega_{10}$. To get the frequency in Hz,

$$\nu_{10} = c/(2a) \tag{4.1}$$
$$= (3 \times 10^8)/(2 \times 2.286 \times 10^{-2}) \tag{4.2}$$
$$\nu_{10} = 6.56 \text{ GHz}, \tag{4.3}$$
$$\therefore \quad \nu_{01} = 13.1 \text{ GHz}. \tag{4.4}$$

4–2 Find the electromagnetic field energy per unit length of waveguide for TE_{10} mode. Express your result in terms of $ab\varepsilon_0 E_{\max}^2$. Start with the fields for the TE_{10} mode

$$E_y = E_{\max} \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t), \tag{4.5}$$

$$B_x = -E_{\max} \frac{k}{\omega} \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t), \tag{4.6}$$

$$B_z = -E_{\max} \frac{\pi}{a\omega} \cos\left(\frac{\pi x}{a}\right) \cos(kz - \omega t). \tag{4.7}$$

where $k = \sqrt{\omega^2/c^2 - (\pi/a)^2}$.

Solution

The time-averaged energy in the EM field per unit length of the waveguide is

$$\frac{U}{L} = \int_0^a \int_0^b \frac{\varepsilon_0}{2} \left\langle [E_y(\mathbf{r},t)]^2 \right\rangle dy\, dx + \int_0^a \int_0^b \frac{1}{2\mu_0} \left\{ \left\langle [B_x(\mathbf{r},t)]^2 \right\rangle + \left\langle [B_z(\mathbf{r},t)]^2 \right\rangle \right\} dy\, dx, \tag{4.8}$$

$$= \int_0^a \int_0^b \frac{\varepsilon_0}{2} \left\langle E_{\max}^2 \sin^2\left(\frac{\pi x}{a}\right) \sin^2(kz-\omega t) \right\rangle dy\, dx + \int_0^a \int_0^b \frac{1}{2\mu_0} \left\{ \left\langle E_{\max}^2 \frac{k^2}{\omega^2} \sin^2\left(\frac{\pi x}{a}\right) \sin^2(kz-\omega t) \right\rangle + \left\langle E_{\max}^2 \frac{\pi^2}{a^2\omega^2} \cos^2\left(\frac{\pi x}{a}\right) \cos^2(kz-\omega t) \right\rangle \right\} dy\, dx. \tag{4.9}$$

Now, we can use the following:

$$\left\langle \sin^2(kz-\omega t) \right\rangle = \left\langle \cos^2(kz-\omega t) \right\rangle = \frac{1}{2}; \quad \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx = \int_0^a \cos^2\left(\frac{\pi x}{a}\right) dx = \frac{a}{2}. \tag{4.10}$$

Then

$$\frac{U}{L} = \frac{\varepsilon_0}{2}\frac{ab}{4} E_{\max}^2 + \frac{1}{2\mu_0}\frac{ab}{4}\frac{k^2}{\omega^2} E_{\max}^2 + \frac{1}{2\mu_0}\frac{ab}{4}\frac{\pi^2}{a^2\omega^2} E_{\max}^2, \tag{4.11}$$

and using the dispersion relation given we can substitute for k above to get

$$\frac{U}{L} = \frac{\varepsilon_0}{2}\frac{ab}{4} E_{\max}^2 + \frac{1}{2\mu_0}\frac{ab}{4}\left(\frac{1}{c^2} - \frac{\pi^2}{a^2\omega^2}\right) E_{\max}^2 + \frac{1}{2\mu_0}\frac{ab}{4}\frac{\pi^2}{a^2\omega^2} E_{\max}^2, \tag{4.12}$$

$$= \frac{\varepsilon_0}{2}\frac{ab}{4} E_{\max}^2 + \frac{1}{2\mu_0}\frac{ab}{4}\frac{1}{c^2} E_{\max}^2. \tag{4.13}$$

Hence, remembering that $\mu_0\varepsilon_0 = 1/c^2$, the energy per unit length of waveguide for TE_{10} mode is

$$\frac{U}{L} = \frac{\varepsilon_0 ab}{4} E_{\max}^2. \tag{4.14}$$

4–3 Consider a hollow standard WR-159 F-band waveguide which has an internal cross section of 40.386 mm $\times$ 20.193 mm.

(a) A 2 GHz signal is fed into the waveguide. Will this cause a wave to propagate along the waveguide?

(b) Assuming the wave would be in the TE_{10} mode, what is the wavenumber? Discuss the implications of your answer.

Solution

(a) The lowest cutoff frequency is for the TE_{10} with $\omega_{10} = c\pi/a = 2.33 \times 10^{10}$ Hz, hence $\nu_{10} = c/2a = 3.71$ GHz. 2 GHz is below the cut-off frequency for the TE_{10} mode, and so the wave will decay exponentially with distance.

(b) The wavenumber is obtained from the dispersion relation with $\omega = 2\pi \times 2$ rad s^{-1}

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{10}^2} = \pm i\, 65.6 \quad \mathrm{m}^{-1}. \tag{4.15}$$

Because this is purely imaginary the wave will not propagate, and the electromagnetic field will decay exponentially as $e^{-\mathrm{Im}\{k\}z}$ if the wave was launched in the $+\widehat{\mathbf{z}}$ direction. The field attenuation length is $1/\mathrm{Im}\{k\} = 15.3$ mm.

4–4 Consider the case of launching an EM wave with frequency $\omega < \omega_{10}$ into a rectangular waveguide. Assuming the wave will attempt to propagate as a TE_{10} wave travelling in along the waveguide (in the z direction),
(a) Derive all the components of the electromagnetic field for the case where k is purely imaginary.
(b) Obtain the Poynting vector and discuss the energy flow.

Solution

(a) We will need to obtain the (real) EM field $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ of the TE_{10} mode for the case where k is purely imaginary. We know that for all the TE modes $E_z^0(x,y) = 0$ and that the amplitude function of the longitudinal component of the magnetic field for the TE_{10} mode is

$$B_z^0(x,y) = B_{10}\cos\left(\frac{\pi x}{a}\right). \tag{4.16}$$

We can get all the other field components from B_z^0 and E_z^0 using

$$B_x^0(x,y) = \left(k\frac{\partial B_z^0}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z^0}{\partial y}\right)\beta \;=\; -\beta k B_{10}^0\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right), \tag{4.17}$$

$$B_y^0(x,y) = \left(k\frac{\partial B_z^0}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z^0}{\partial x}\right)\beta \;=\; 0, \tag{4.18}$$

$$E_x^0(x,y) = \left(k\frac{\partial E_z^0}{\partial x} + \omega\frac{\partial B_z^0}{\partial y}\right)\beta \;=\; 0, \tag{4.19}$$

$$E_y^0(x,y) = \left(k\frac{\partial E_z^0}{\partial y} - \omega\frac{\partial B_z^0}{\partial x}\right)\beta \;=\; +\omega\beta B_{10}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi x}{a}\right), \tag{4.20}$$

where we have used

$$\beta \equiv \frac{ic^2}{\omega^2 - k^2c^2} \;=\; \frac{ic^2}{\omega_{10}^2} \;=\; \frac{ic^2}{(\pi c/a)^2} \;=\; \frac{i\,a^2}{\pi^2}. \tag{4.21}$$

We need to multiply these field amplitude functions by $\exp\left[i\left(k_{10}z(\omega) - \omega t\right)\right]$ to get the (complex) EM field of a TE_{10} wave in the z-direction:

$$E_x(\mathbf{r},t) = 0, \tag{4.22}$$

$$E_y(\mathbf{r},t) = +\,B_{10}\,i\,\omega\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}, \tag{4.23}$$

$$E_z(\mathbf{r},t) = 0, \tag{4.24}$$

$$B_x(\mathbf{r},t) = -\,B_{10}\,i\,k\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}, \tag{4.25}$$

$$B_y(\mathbf{r},t) = 0, \tag{4.26}$$

$$B_z(\mathbf{r},t) = +\,B_{10}\cos\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}, \tag{4.27}$$

$$\text{with}\quad k = \sqrt{\omega^2/c^2 - (\pi/a)^2}. \tag{4.28}$$

In the present problem, $k(\omega) = i\,\mathrm{Im}\{k(\omega)\}$, and so the (real) EM field of the TE_{10} wave in the z-direction is

$$E_x(\mathbf{r},t) = 0, \tag{4.29}$$

$$E_y(\mathbf{r},t) = \mathrm{Re}\left\{+\,B_{10}\,i\,\omega\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}\right\}, \tag{4.30}$$

$$= -\,B_{10}\,\omega\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\sin(-\omega t), \tag{4.31}$$

$$= E_{\max}\,\sin\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\sin(\omega t), \tag{4.32}$$

$$E_z(\mathbf{r},t) = 0, \tag{4.33}$$

$$B_x(\mathbf{r},t) = \mathrm{Re}\left\{-\,B_{10}\,i\,k\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}\right\}, \tag{4.34}$$

$$= +\,B_{10}\,\mathrm{Im}\{k\}\,\left(\frac{a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\sin(-\omega t), \tag{4.35}$$

$$= -E_{\max}\frac{k}{\omega}\sin\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\sin(\omega t), \tag{4.36}$$

$$B_y(\mathbf{r},t) = 0, \tag{4.37}$$

$$B_z(\mathbf{r},t) = \mathrm{Re}\left\{+\,B_{10}\cos\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}\right\}, \tag{4.38}$$

$$= +\,B_{10}\,\frac{\omega a}{\pi}\cos\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\cos(-\omega t), \tag{4.39}$$

$$= -E_{\max}\frac{\pi}{\omega a}\cos\left(\frac{\pi x}{a}\right)\,e^{-\mathrm{Im}\{k\}\,z}\,\cos(\omega t). \tag{4.40}$$

(b) The Poynting vector is $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H} = (E_y H_z \widehat{\mathbf{x}} - E_y H_x \widehat{\mathbf{z}})$ for this problem, then

$$E_y(\mathbf{r}, t) \;=\; E_{\max} \sin\left(\frac{\pi x}{a}\right) e^{-\mathrm{Im}\{k\} z} \sin(\omega t), \tag{4.41}$$

$$H_x(\mathbf{r}, t) = -E_{\max} \frac{\mathrm{Im}\{k\}}{\omega \mu_0} \sin\left(\frac{\pi x}{a}\right) e^{-\mathrm{Im}\{k\}\, z} \sin(\omega t), \tag{4.42}$$

$$H_z(\mathbf{r}, t) = -E_{\max} \frac{\pi}{\omega \mu_0 a} \cos\left(\frac{\pi x}{a}\right) e^{-\mathrm{Im}\{k\}\, z} \cos(\omega t). \tag{4.43}$$

Hence, the non-zero components of the Poynting vector are

$$S_z(\mathbf{r}, t) \;=\; E_{\max}^2 \frac{\mathrm{Im}\{k\}}{\omega \mu_0} \sin^2\left(\frac{\pi x}{a}\right) e^{-2\mathrm{Im}\{k\} z} \frac{1}{2} \sin(2\omega t). \tag{4.44}$$

$$S_x(\mathbf{r}, t) \;=\; -E_{\max}^2 \frac{\pi}{\omega \mu_0 a} \frac{1}{2} \sin\left(2\frac{\pi x}{a}\right) e^{-2\mathrm{Im}\{k\} z} \frac{1}{2} \sin(2\omega t), \tag{4.45}$$

where we have used $\cos A \sin A = \sin(2A)/2$.

The result is that $\mathbf{S}$ oscillates at frequency 2ω and decays with increasing z, and $\langle \mathbf{S} \rangle = 0$. The intensity attenuation length is $(2\,\mathrm{Im}\{k\})^{-1}$.

5 Radiation and scattering

5–1 A walkie-talkie has a short centre-fed dipole antenna 12 cm long and radiates 3 W at 450 MHz.

(a) What is the radiation resistance of the antenna? Assume the antenna has been balanced by a suitable inductor such that its impedance is purely resistive, and that the transmission line from the transceiver is matched to the impedance of the antenna.

(b) What is the value of the peak current in the antenna?

(c) What is the peak electric field at the location of a receiving antenna at distance r?

(d) With the transmitting walkie-talkie held vertically, an identical receiving walkie-talkie also held vertically at a distance of 2 km away can barely receive the transmission, i.e. the range is 2 km. What would be the range if: (i) the transmitting antenna were tilted at 60° towards the receiving antenna, (ii) the transmitting antenna was vertical but the receiving antenna was tilted at 60° towards the transmitting antenna, (iii) the transmitting antenna was vertical but the receiving antenna was tilted at 60° to the vertical in a plane perpendicular to the line of sight to the transmitting walkie-talkie?

Solution

(a) The wavelength is $\lambda = 3 \times 10^8 / 4.5 \times 10^8 = 0.667$ m, giving $d/\lambda = 0.18$ and so we shall use formulae relevant to a short ($d \ll \lambda$) centre-fed dipole antenna. The radiation resistance of this short centre-fed dipole antenna is then approximately

$$R_{\mathrm{rad}} = 197 \left(\frac{d}{\lambda}\right)^2 = 197 \times 0.18^2 = 6.38\,\Omega. \tag{5.1}$$

(b) We can use the time-averaged power $\langle P \rangle = 3$ W, together with the radiation resistance, to obtain the peak current I_0 as follows

$$I_0^2 = 2\langle I^2 \rangle = 2\frac{\langle P \rangle}{R_{\mathrm{rad}}}, \qquad \therefore \quad I_0 = \sqrt{\frac{2 \times 3}{6.38}} = 0.97\,\mathrm{A}. \tag{5.2}$$

(c) The antenna will emit electric dipole radiation, and for this short centre-fed dipole antenna the dipole's amplitude is

$$p_0 = \frac{I_0 d}{2\omega} = \frac{0.97 \times 0.12}{2 \times (2\pi \times 4.5 \times 10^8)} = 2.06 \times 10^{-11}\,\mathrm{C\ \ m}. \tag{5.3}$$

For electric dipole radiation the electric field, and field amplitude, are

$$\mathbf{E}(\mathbf{r}, t) = -\left(\frac{k^2}{4\pi\varepsilon_0}\right) p_0 \frac{e^{i(kr-\omega t)}}{r} \sin\theta\, \widehat{\boldsymbol{\theta}}. \tag{5.4}$$

$$\therefore\ \mathbf{E}_0 = -\left(\frac{\omega^2}{4\pi\varepsilon_0 c^2}\right) \frac{p_0}{r} \sin\theta\, \widehat{\boldsymbol{\theta}}, \tag{5.5}$$

$$= -\frac{(2\pi \times 4.5 \times 10^8)^2}{4\pi \times (8.85 \times 10^{-12}) \times (3 \times 10^8)^2}\, 2.06 \times 10^{-11} \frac{\sin\theta}{r} \widehat{\boldsymbol{\theta}}. \tag{5.6}$$

$$= -16.5 \frac{\sin\theta}{r} \widehat{\boldsymbol{\theta}} \quad (\text{V m}^{-1}). \tag{5.7}$$

(d) When both antennas are vertical, in a horizontal plane, and 2 km apart, the electric field component parallel to the receiving antenna is the minimum detectable,

$$E^{\|}_{\text{min}} = \frac{16.5}{2000} \sin 90^\circ = 8.25 \times 10^{-3} \text{ V m}^{-1}. \tag{5.8}$$

(i) For the transmitting antenna tilted by 60° towards the receiving antenna we have $\theta = 60^\circ$ in Eq. 5.7, and $\widehat{\boldsymbol{\theta}}$ is pointing vertically down (parallel to the receiving dipole antenna). Hence, at the detection threshold we must have

$$E^{\|}_{\text{min}} = 16.5 \frac{\sin 60^\circ}{r}, \qquad \therefore\ r = 1.73 \text{ km}. \tag{5.9}$$

(ii) If the transmitting antenna is vertical and the receiving antenna is tilted by 60° towards the transmitting antenna, the electric field is vertical but the component parallel to the antenna is reduced by $\cos 60^\circ$. Hence

$$E^{\|}_{\text{min}} = 16.5 \frac{\sin 90^\circ\ \cos 60^\circ}{r}, \qquad \therefore\ r = 1.00 \text{ km}. \tag{5.10}$$

(iii) If the transmitting antenna is vertical and the receiving antenna is tilted by 60° from the vertical perpendicular the direction to the transmitting antenna, the electric field is vertical but the component parallel to the antenna is reduced to $E_0 \cos 60^\circ$. Hence, as before,

$$E^{\|}_{\text{min}} = 16.5 \frac{\sin 90^\circ\ \cos 60^\circ}{r}, \qquad \therefore\ r = 1.00 \text{ km}. \tag{5.11}$$

5–2 Starting with the Clausius-Mossotti formula and the time-averaged dipole radiation power

$$\alpha_m = \frac{\varepsilon_0}{n_v}(\varepsilon_r - 1), \qquad \langle P \rangle = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3}, \tag{5.12}$$

find the cross section for scattering of light at $\lambda_{500} \times 500$ nm wavelength by air molecules, and its associated mean free path $1/(n_v\sigma)$. Dry air at STP has $\rho = 1.30$ kg m^{-3}, and $n = 1.00029$, and its molecular weight is 29.

Solution

A molecule acquires an electric dipole moment in the presence of an electric field $\mathbf{p} = \alpha_m \mathbf{E}$. For the case of a dilute gas the Clausius-Mossotti equation yields

$$\alpha_m = \frac{\varepsilon_0}{n_v}(\varepsilon_r - 1) = \frac{\varepsilon_0}{n_v}(n^2 - 1) \tag{5.13}$$

where n_v is the number density of molecules, ε_r the relative permittivity and n is the refractive index.

In the presence of an oscillating electric field due to an incident monochromatic plane wave $\mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$ the molecule becomes an oscillating dipole which radiates with time-averaged power

$$\langle P \rangle = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3} = \left(\frac{\varepsilon_0}{n_v}\right)^2 \frac{(n^2-1)^2 E_0^2 \omega^4}{12\pi\varepsilon_0 c^3}. \tag{5.14}$$

Dividing $\langle P \rangle$ by the time-averaged energy flux of the incident wave $\langle S \rangle = \frac{1}{2}\varepsilon_0 E_0^2 c$, we obtain the cross section

$$\sigma_{\mathrm{mol}} = \frac{8\pi^3}{3} \frac{(n^2-1)^2}{n_v^2} \lambda^{-4}. \tag{5.15}$$

From the data on dry air given,

$$(n^2-1)^2 = 3.4 \times 10^{-7}, \tag{5.16}$$

$$n_v = \frac{\rho}{29u} = \frac{1.30}{29 \times 1.66 \times 10^{-27}} = 2.6 \times 10^{25} \quad \mathrm{m}^{-3}. \tag{5.17}$$

$$\therefore \quad \sigma(\lambda) = \frac{8\pi^3}{3} \frac{(n^2-1)^2}{n_v^2} \lambda^{-4} \tag{5.18}$$

$$= \frac{8\pi^3}{3} \frac{3.4 \times 10^{-7}}{(2.6 \times 10^{25})^2} \times (500 \times 10^{-9} \lambda_{500})^{-4}, \tag{5.19}$$

$$= 6.65 \times 10^{-31} \lambda_{500}^{-4} \ \mathrm{m}^2. \tag{5.20}$$

The mean free path is $1/n_v\sigma = 58\lambda_{500}^4$ km, where $\lambda_{500} = \lambda/(500 \text{ nm})$.

5–3 In the presence of an applied uniform electric field $\mathbf{E} = E_0 \widehat{\mathbf{z}}$ the radial component of the electric field outside a perfectly-conducting sphere of radius a centred at the origin becomes

$$E_r(r, \theta, \phi) = E_0 \left(1 + 2\frac{a^3}{r^3}\right) \cos\theta. \tag{5.21}$$

Find the cross section for scattering of monochromatic EM waves by a perfectly-conducting sphere of radius: (a) $a \ll \lambda$, and (b) $a \gg \lambda$. Assume electric dipole radiation is responsible.

Solution

(a) For a monochromatic plane wave with electric field $\mathbf{E}(\mathbf{r}, t) = E_0 \cos(kx - \omega t)\widehat{\mathbf{z}}$ with

$k^{-1} \gg a$, the applied electric field over the entire sphere can be approximated by $\mathbf{E}(a,\theta,\phi,t) \approx E_0\cos(\omega t)\widehat{\mathbf{z}}$.

Then, using Gauss' law, the surface charge density is

$$\sigma(a,\theta,\phi,t) \;=\; \varepsilon_0\mathbf{E}(a,\theta,\phi,t)\cdot\widehat{\mathbf{r}} \;=\; \varepsilon_0 E_r(a,\theta,\phi,t) \;=\; 3\varepsilon_0 E_0\cos(\omega t)\cos\theta. \tag{5.22}$$

The dipole moment will be in the $\widehat{\mathbf{z}}$ direction, and its z component will be

$$p_z(t) \;=\; \widehat{\mathbf{z}}\cdot\oint \mathbf{r}\,\sigma(\mathbf{r},t)\,dS, \tag{5.23}$$

$$\;=\; \cos(\omega t)\int(a\cos\theta)(3\varepsilon_0 E_0\cos\theta)\,2\pi a^2\,d(\cos\theta), \tag{5.24}$$

$$\therefore\;\; p_z(t) \;=\; 6\pi a^3\varepsilon_0 E_0\cos(\omega t)\int_{-1}^{1}\cos^2\theta\,d(\cos\theta). \tag{5.25}$$

$$\therefore\;\; \mathbf{p}_z(t) \;=\; 4\pi a^3\varepsilon_0 E_0\cos(\omega t)\,\widehat{\mathbf{z}}. \tag{5.26}$$

Using Larmor's formula for electric dipole radiation, the time-averaged power is

$$\langle P\rangle \;=\; \frac{p_0^2\,\omega^4}{12\pi\varepsilon_0 c^3} \;=\; \frac{\left(4\pi a^3\varepsilon_0\right)^2 E_0^2\,\omega^4}{12\pi\varepsilon_0 c^3} \;=\; \frac{4\pi a^6\varepsilon_0 E_0^2\,\omega^4}{3c^3}. \tag{5.27}$$

Because the dipole radiation pattern is proportional to $\sin^2\theta$ where θ is the angle to the acceleration direction, the radiation will be zero in the direction of acceleration, i.e. in the $\pm\widehat{\mathbf{z}}$ direction and maximum at all directions in the x—y plane.

The time-averaged Poynting vector for the EM wave is

$$\langle\mathbf{S}(\mathbf{r})\rangle \;=\; \frac{1}{2}\varepsilon_0 E_0^2 c\,\widehat{\mathbf{z}}. \tag{5.28}$$

Hence, the scattering cross section due to electric dipole radiation is

$$\sigma_{\rm sc}^{EDR} \;=\; \frac{\langle P\rangle}{\langle S\rangle} \;=\; \left(\frac{4\pi a^6\varepsilon_0 E_0^2\,\omega^4}{3c^3}\right)\Big/\left(\frac{1}{2}\varepsilon_0 E_0^2 c\right) \;=\; \frac{8\pi a^6}{3}\frac{\omega^4}{c^4} \;=\; \frac{8\pi}{3}(k^4a^4)a^2. \tag{5.29}$$

Since the wavenumber has units (m^{-1}) and a has units (m), the cross section has units (m^2) as required. The scattering cross section is proportional to ω^4 which is the same dependence as for scattering by molecules which was found by Rayleigh.

Note that the cross section we have calculated is actually incomplete as it neglects magnetic dipole radiation from oscillating surface current loops induced by the changing magnetic field of the incident electromagnetic wave. Including magnetic dipole radiation would increase the calculated cross section by 25%.

(b) The cross section for the case where $k^{-1} \ll a$, is for wavelengths $\lambda \ll a$, in which case the perfectly conducting sphere may be considered a perfect reflector blocking out its cross-sectional area $\sigma = \pi a^2$.

5–4 In the presence of an applied uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ a perfectly-conducting sphere (radius a) acquires a magnetic dipole moment $\mathbf{m}_0$.
(a) What are the boundary conditions at the surface of the sphere? Find the magnetic dipole moment that, together with the applied magnetic field, satisfies the boundary conditions.
(b) Find the contribution of magnetic dipole radiation to the cross section for scattering of monochromatic EM waves by a perfectly-conducting sphere of radius $a \ll \lambda$.

Solution

(a) Inside a perfect conductor $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$, so that the boundary conditions on the electromagnetic fields at the conductor's surface just outside the conductor are simplified to $\mathbf{E}_\parallel = \mathbf{0}$ and $\mathbf{B}_\perp = \mathbf{0}$. That is, the electric field at the surface is normal to the surface, and the magnetic field at the surface is parallel to the surface.

(b) Adding the field of a magnetic dipole at the origin with moment $\mathbf{m} = \mathbf{m}_0\widehat{\mathbf{z}}$ to the constant field $B_0\widehat{\mathbf{z}}$,

$$\mathbf{B}(\mathbf{r}) = B_0\,\widehat{\mathbf{z}} + \frac{\mu_0}{4\pi}\left[\frac{3\,\widehat{\mathbf{r}}\,m_0\cos\theta - m_0\widehat{\mathbf{z}}}{r^3}\right], \tag{5.30}$$

$$= B_0(\cos\theta\,\widehat{\mathbf{r}} - \sin\theta\,\widehat{\boldsymbol{\theta}}\,) + \frac{\mu_0 m_0}{4\pi}\left[\frac{3\,\widehat{\mathbf{r}}\,\cos\theta - (\cos\theta\,\widehat{\mathbf{r}} - \sin\theta\,\widehat{\boldsymbol{\theta}}\,)}{r^3}\right]. \tag{5.31}$$

At the surface of the conductor $B_r(a,\theta,\phi)$ must be zero, and so

$$B_0\cos\theta + \frac{\mu_0 m_0}{4\pi a^3}2\cos\theta = 0. \tag{5.32}$$

$$\therefore\quad \mathbf{m}_0 = -\frac{2\pi a^3}{\mu_0}B_0\widehat{\mathbf{z}}. \tag{5.33}$$

For a monochromatic plane wave with magnetic field $\mathbf{B}(\mathbf{r},t) = B_0\cos(kx-\omega t)\,\widehat{\mathbf{z}}$ with $k^{-1} \gg a$, the applied magnetic field near the conductor can be approximated by $\mathbf{B}(\mathbf{r},t) \approx B_0\cos(\omega t)\,\widehat{\mathbf{z}}$. Hence,

$$\mathbf{m}(t) = \mathbf{m}_0\cos(\omega t)\,\widehat{\mathbf{z}}. \tag{5.34}$$

For an oscillating magnetic dipole, the time-averaged radiation power is

$$\langle P\rangle = \frac{\mu_0\omega^4 m_0^2}{12\pi c^3} = \frac{\mu_0\omega^4}{12\pi c^3}\frac{4\pi^2 a^6}{\mu_0^2}B_0^2. \tag{5.35}$$

The time-averaged Poynting vector for the EM wave is

$$\langle\mathbf{S}(\mathbf{r})\rangle \;=\; \frac{1}{2\mu_0}B_0^2 c\,\widehat{\mathbf{z}}. \tag{5.36}$$

Hence, the contribution of magnetic dipole radiation to the scattering cross section is

$$\sigma_{\text{sc}}^{MDR} \;=\; \frac{\langle P\rangle}{\langle S\rangle} \;=\; \left(\frac{\mu_0\omega^4}{12\pi c^3}\frac{4\pi^2 a^6}{\mu_0^2}B_0^2\right)\Big/\left(\frac{1}{2\mu_0}B_0^2 c\right) \;=\; \frac{2\pi}{3}(k^4a^4)a^2. \tag{5.37}$$

Adding this to the contribution of electric dipole radiation to the scattering cross section

(Exercise 5–3) we obtain

$$\sigma_{\rm sc}^{\rm tot} = \sigma_{\rm sc}^{MDR} + \sigma_{\rm sc}^{EDR} = \frac{10\pi}{3}(k^4a^4)a^2. \tag{5.38}$$

5–5 In bremsstrahlung an energetic electron is accelerated in the electric field of an atomic nucleus and emits electromagnetic radiation. Consider the case of an electron of kinetic energy 10 keV travelling on a trajectory parallel to the z-axis, and that there is a lead (Pb) nucleus located at impact parameter $b = 10^{-10}$ m (distance of closest approach to the initial trajectory). Assume that for this impact parameter the electron's deflection is small, and that a calculation of the energy lost to electromagnetic radiation can be done classically.

(a) By comparing b with the de Broglie wavelength h/mv, reassure yourself that a classical calculation is valid for this impact parameter and velocity.

(b) By checking whether or not the deflection angle is small, reassure yourself that approximating the trajectory by a straight-line is valid.

(c) Estimate the total energy radiated by the electron. [Neglect atomic electrons.]

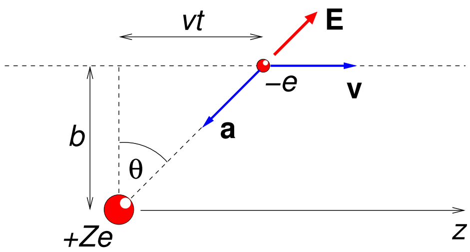
Solution

(a) From the kinetic energy, $E_K = (1/2)mv^2$, we first find the electron's velocity, and then the de Broglie wavelength

$$v = \sqrt{\frac{2E_K}{m_e}} = \sqrt{\frac{2\times(10^4\times 1.6\times 10^{-19})}{9.11\times 10^{-31}}} = 5.93\times 10^7 \text{ m s}^{-1}, \tag{5.39}$$

$$\lambda_{dB} = \frac{h}{mv} = \frac{6.62\times 10^{-34}}{(9.11\times 10^{-31})\times(5.93\times 10^7)} = 1.23\times 10^{-11} \text{ m}. \tag{5.40}$$

The impact parameter $b = 10^{-10}$ m is about a factor of 10 larger than the de Broglie wavelength, and so use of classical physics is (marginally) justified.



(b) The straight line approximation is only valid when the impulse is small compared to the initial momentum, i.e. $\Delta p_\perp \ll m_e v$ where

$$\begin{aligned}
\Delta p_\perp &= \int_{-\infty}^{\infty} \frac{Ze^2}{4\pi\varepsilon_0(b^2+v^2t^2)}\cos[\theta(t)]dt, && (5.41)\\
&= \int_{-\infty}^{\infty} \frac{Ze^2}{4\pi\varepsilon_0}\frac{b}{(b^2+v^2t^2)^{3/2}}dt, && (5.42)\\
&= \frac{Ze^2}{2\pi\varepsilon_0 bv}. && (5.43)
\end{aligned}$$

Since the straight line approximation is valid when $\Delta p_\perp \ll m_e v$, we require

$$b \gg \frac{Ze^2}{2\pi\varepsilon_0 m_e v^2} = 1.18\times 10^{-11}\ \mathrm{m} \qquad (5.44)$$

which is marginally satisfied since $b = 10^{-10}$ m.

(c) Larmor's formula for the instantaneous radiation power is

$$P(t) = \frac{a(t)^2e^2}{6\pi\varepsilon_0 c^3}, \qquad (5.45)$$

and so the total energy radiated by the electron as it travels from $z = -\infty$ to $z = +\infty$ is

$$
\begin{aligned}
W &= \frac{e^2}{6\pi\varepsilon_0 c^3}\int_{-\infty}^{\infty} a(t)^2 dt, && (5.46)\\
&= \frac{e^2}{6\pi\varepsilon_0 c^3}\int_{-\infty}^{\infty}\left(\frac{(Ze)\times e}{4\pi\varepsilon_0(b^2+v^2t^2)m_e}\right)^2 dt, && (5.47)\\
&= \frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{Ze^2}{4\pi\varepsilon_0 m_e}\right)^2\int_{-\infty}^{\infty}\frac{1}{(b^2+v^2t^2)^2}dt, && (5.48)\\
&= \frac{Z^2e^6}{96\pi^3\varepsilon_0^3c^3m_e^2}\left[\frac{bt}{2b^3(b^2+v^2t^2)}+\frac{\tan^{-1}(vt/b)}{2b^3v}\right]_{-\infty}^{\infty}, && (5.49)\\
&= \frac{Z^2e^6}{96\pi^3\varepsilon_0^3c^3m_e^2}\frac{\pi}{2b^3v}, && (5.50)\\
&= \frac{Z^2e^6}{192\pi^2\varepsilon_0^3c^3m_e^2b^3v}, && (5.51)\\
&= 4.04\times 10^{-5}\ \text{eV}. && (5.52)
\end{aligned}
$$

5–6 A pulsar is a rapidly-spinning magnetised neutron star observed at radio and sometimes higher (optical, X-ray and gamma-ray) frequencies. Neutron stars have masses of typically $M \sim 1.4\ \mathrm{M}_\odot$ (solar mass $1\ \mathrm{M}_\odot \approx 2\times 10^{30}$ kg), radius $R \approx 10$ km, moment of inertia $I_{MoI} = \frac{2}{5}MR^2 \approx 10^{38}$ kg m^2 and a wide range of surface magnetic fields. Outside the neutron star the magnetic field may be approximated by that of a magnetic dipole at its centre making angle α to the spin axis. Pulsars typically emit one or more narrow pulses at the same time into each spin period P, and measurements at different epochs generally show that the spin is slowing down as characterised by the time-derivative of the period $\dot{P}$.

(a) Derive formulae in terms of P and $\dot{P}$ for the rotational kinetic energy $E_{\rm rot}$, the rate of loss of rotational kinetic energy $\dot{E}_{\rm rot}$, the pulsar's characteristic age $\tau = E_{\rm rot}/\dot{E}_{\rm rot}$ and, assuming the slow-down is due to conversion of rotational kinetic energy to magnetic dipole radiation, the pulsar's minimum equatorial surface magnetic field $B_{\rm min}$.

(b) The radio pulsar PSR J0157+6212 has $P = 2.355$ s and $\dot{P} = 1.89\times 10^{-13}$ ("The Australia Telescope National Facility Pulsar Catalogue", Manchester, R.N., et al., 2005, Astron. J., 129, 1993, and http://www.atnf.csiro.au/research/pulsar/psrcat/). Find $E_{\rm rot}$, $\dot{E}_{\rm rot}$, $\tau = E_{\rm rot}/\dot{E}_{\rm rot}$ and $B_{\rm min}$.

Solution

(a) The rotational kinetic energy is

$$E_{\mathrm{rot}} \;=\; \frac{1}{2} I_{MoI}\omega^2 \;=\; \frac{1}{2} I_{MoI}\left(\frac{2\pi}{P}\right)^2 \;=\; 2\pi^2 I_{MoI} P^{-2}. \tag{5.53}$$

The rate of loss of rotational kinetic energy is

$$\dot{E}_{\mathrm{rot}} \;=\; 2\pi^2 I_{MoI}(-2)P^{-3}\dot{P} \;=\; -4\pi^2 I_{MoI}\frac{\dot{P}}{P^3}\,. \tag{5.54}$$

The pulsar's characteristic age is

$$\tau \;=\; \frac{2\pi^2 I_{MoI} P^{-2}}{4\pi^2 I_{MoI}\dot{P}P^{-3}} \;=\; \frac{P}{2\dot{P}}\,. \tag{5.55}$$

For the pulsar, we have a rotating magnetic dipole of effective moment $m_0 \sin\alpha$, where α is the angle between the pulsar's magnetic moment and its spin axis as it is only the moment's component perpendicular to the spin axis which is relevant for radiation. Furthermore, a rotating magnetic dipole moment is equivalent to two orthogonal oscillating magnetic dipole moments out of phase by 90°.

We need to write the magnetic moment in terms of the surface magnetic field, and for a magnetic dipole of moment $\mathbf{m}_0$

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{r}(\mathbf{m_0}\cdot\mathbf{r}) - r^2\mathbf{m_0}}{r^5}\right], \tag{5.56}$$

and so at the magnetic equator of a neutron star ($\mathbf{m_0}\cdot\mathbf{r} = 0$ in the formula above) the surface magnetic field has magnitude

$$B \;=\; \frac{\mu_0}{4\pi}\frac{m_0}{R^3}, \qquad \therefore\;\; m_0 \;=\; \frac{4\pi}{\mu_0}R^3 B. \tag{5.57}$$

The time-averaged magnetic dipole radiation power of a rotating dipole is twice the power of a single oscillating magnetic dipole

$$\langle P_{\mathrm{MDR}}\rangle \;=\; 2\times\frac{\mu_0\omega^4(m_0\sin\alpha)^2}{12\pi c^3}. \tag{5.58}$$

Substituting for the magnetic dipole moment in terms of the polar surface magnetic field we get

$$\langle P_{\mathrm{MDR}} \rangle = \frac{8\pi^3 \mu_0}{3c^3 P^4} \left(\frac{4\pi}{\mu_0} R^3 B \sin\alpha \right)^2 . \tag{5.59}$$

Equating this to the rate of loss of rotational kinetic energy is

$$\frac{8\pi^3 \mu_0}{3c^3 P^4} \left(\frac{4\pi}{\mu_0} R^3 B \sin\alpha \right)^2 = 4\pi^2 I_{MoI} \frac{\dot{P}}{P^3}, \tag{5.60}$$

$$B^2 \sin^2\alpha = 4\pi^2 I_{MoI} \frac{\dot{P}}{P^3} \frac{3c^3 P^4}{8\pi^3 \mu_0} \frac{\mu_0^2}{16\pi^2 R^6}. \tag{5.61}$$

$$\therefore \quad B \sin\alpha = \left(\frac{3c^3 \mu_0 I_{MoI}}{32\pi^3 R^6} \right)^{1/2} \left(\dot{P} P \right)^{1/2} . \tag{5.62}$$

$$\therefore \quad B > 3.20 \times 10^{15} \left(\dot{P} P \right)^{1/2} \quad (\mathrm{T}). \tag{5.63}$$

(b) For pulsar PSR J0157+6212 the rotational kinetic energy is

$$E_{\rm rot} \;=\; 2\pi^2 \times 10^{38} \times P^{-2} \;=\; 3.56 \times 10^{38}\ \text{J}. \tag{5.64}$$

The rate of loss of rotational kinetic energy is

$$\dot{E}_{\rm rot} \;=\; 4\pi^2 \times 10^{38}\,\frac{\dot{P}}{P^3} \;=\; 5.71 \times 10^{25}\ \text{W}. \tag{5.65}$$

The pulsar's characteristic age is

$$\tau \;=\; \frac{P}{2\dot{P}} \;=\; 6.2 \times 10^{12}\ \text{s} \;=\; 2.0 \times 10^{5}\ \text{y}. \tag{5.66}$$

The pulsar's minimum polar surface magnetic field is

$$B_{\rm min} \;=\; 3.20 \times 10^{15} \left(\dot{P}P\right)^{1/2} \;=\; 2.14 \times 10^{9}\ \text{(T)}. \tag{5.67}$$