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Portfolio Theory & Financial Analyses

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About the Author

With an eclectic record of University teaching, research, publication, consultancy and curricula development, underpinned by running a successful business, Alan has been a member of national academic validation bodies and held senior external examinerships and lectureships at both undergraduate and postgraduate level in the UK and abroad.

With increasing demand for global e-learning, his attention is now focussed on the free provision of a financial textbook series, underpinned by a critique of contemporary capital market theory in volatile markets, published by bookboon.com.

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Part I:
An Introduction

1 An Overview

Introduction

Once a company issues shares (common stock) and receives the proceeds, it has no *direct* involvement with their subsequent transactions on the capital market, or the price at which they are traded. These are matters for negotiation between existing shareholders and prospective investors, based on their own financial agenda.

As a basis for negotiation, however, the company plays a pivotal *agency* role through its implementation of investment-financing strategies designed to maximise profits and shareholder wealth. What management do to satisfy these objectives and how the market reacts are ultimately determined by the law of supply and demand. If corporate returns exceed market expectations, share price should rise (and vice versa).

But in a world where ownership is divorced from control, characterised by economic and geo-political events that are also beyond management's control, this invites a question.

> How do companies determine an optimum portfolio of investment strategies that satisfy a multiplicity of shareholders with different wealth aspirations, who may also hold their own diverse portfolio of investments?

1.1 The Development of Finance

As long ago as 1930, Irving Fisher's *Separation Theorem* provided corporate management with a lifeline based on what is now termed Agency Theory.

He acknowledged implicitly that whenever ownership is divorced from control, direct communication between management (*agents*) and shareholders (*principals*) let alone other stakeholders, concerning the likely profitability and risk of every corporate investment and financing decision is obviously impractical. If management were to implement optimum strategies that satisfy each shareholder, the company would also require prior knowledge of every investor's stock of wealth, dividend preferences and risk-return responses to their strategies.

According to Fisher, what management therefore, require is a model of *aggregate* shareholder behaviour. A theoretical abstraction of the real world based on simplifying assumptions, which provides them with a methodology to communicate a diversity of corporate wealth maximising decisions.

To set the scene, he therefore assumed (not unreasonably) that all investor behaviour (including that of management) is *rational* and *risk averse*. They prefer high returns to low returns but less risk to more risk. However, risk aversion does not imply that rational investors will not take a chance, or prevent companies from retaining earnings to gamble on their behalf. To accept a higher risk they simply require a commensurately higher return, which Fisher then benchmarked.

> Management's minimum rate of return on incremental projects financed by retained earnings should equal the return that existing shareholders, or prospective investors, can earn on investments of equivalent risk elsewhere.

He also acknowledged that a company's acceptance of projects internally financed by retentions, rather than the capital market, also denies shareholders the opportunity to benefit from current dividend payments. Without these, individuals may be forced to sell part (or all) of their shareholding, or alternatively borrow at the market rate of interest to finance their own preferences for consumption (income) or investment elsewhere.

To circumvent these problems Fisher assumed that if capital markets are *perfect* with no barriers to trade and a free flow of information (more of which later) a firm's *investment* decisions can not only be *independent* of its shareholders' *financial* decisions but can also satisfy their wealth maximisation criteria.

In Fisher's perfect world:

- Wealth maximising firms should determine optimum *investment* decisions by *financing* projects based on their *opportunity* cost of capital.
- The opportunity *cost* equals the *return* that existing shareholders, or prospective investors, can earn on investments of equivalent risk elsewhere.
- Corporate projects that earn rates of return less than the opportunity cost of capital should be rejected by management. Those that yield equal or superior returns should be accepted.
- Corporate earnings should therefore be distributed to shareholders as dividends, or retained to fund new capital investment, depending on the relationship between project profitability and capital cost.
- In response to rational managerial dividend-retention policies, the final consumption-investment decisions of rational shareholders are then determined independently according to their personal preferences.
- In perfect markets, individual shareholders can always borrow (lend) money at the market rate of interest, or buy (sell) their holdings in order to transfer cash from one period to another, or one firm to another, to satisfy their income needs or to optimise their stock of wealth.

Activity 1

Based on Fisher's Separation Theorem, share price should rise, fall, or remain stable depending on the inter-relationship between a company's project returns and the shareholders desired rate of return. Why is this?

For detailed background to this question and the characteristics of perfect markets you might care to download "Strategic Financial Management" (both the text and exercises) from bookboon.com and look through their first chapters.

1.2 Efficient Capital Markets

According to Fisher, in perfect capital markets where ownership is divorced from control, the separation of corporate dividend-retention decisions and shareholder consumption-investment decisions is not problematical. If management select projects using the shareholders' desired rate of return as a cut-off rate for investment, then at worst corporate wealth should stay the same. And once this information is communicated to the outside world, share price should not fall.

Of course, the Separation Theorem is an abstraction of the real world; a model with questionable assumptions. Investors do not always behave rationally (some speculate) and capital markets are not perfect. Barriers to trade do exist, information is not always freely available and not everybody can borrow or lend at the same rate. But instead of asking whether these assumptions are divorced from reality, the relevant question is whether the model provides a sturdy framework upon which to build.

Certainly, theorists and analysts believed that it did, if Fisher's impact on the subsequent development of finance theory and its applications are considered. So much so, that despite the recent global financial meltdown (or more importantly, because the events which caused it became public knowledge) it is still a basic tenet of finance taught by Business Schools and promoted by other vested interests world-wide (including governments, financial institutions, corporate spin doctors, the press, media and financial web-sites) that:

Capital markets may not be *perfect* but are still reasonably *efficient* with regard to how analysts *process information* concerning corporate activity and how this changes market values once it is conveyed to investors.

An efficient market is one where:

- Information is universally available to all investors at a low cost.
- Current security prices (debt as well as equity) reflect all relevant information.
- Security prices only change when new information becomes available.

Based on the pioneering research of Eugene Fama (1965) which he formalised as the "efficient market hypothesis" (EMH) it is also widely agreed that *information processing efficiency* can take *three forms* based on *two types* of analyses.

The weak form states that *current* prices are determined solely by a *technical* analysis of *past* prices. Technical analysts (or *chartists*) study historical price movements looking for cyclical patterns or trends likely to repeat themselves. Their research ensures that significant movements in current prices relative to their history become widely and quickly known to investors as a basis for immediate trading decisions. Current prices will then move accordingly.

The semi-strong form postulates that current prices not only reflect price history, but all *public* information. And this is where *fundamental analysis* comes into play. Unlike chartists, *fundamentalists* study a company and its business based on historical records, plus its current and future performance (profitability, dividends, investment potential, managerial expertise and so on) relative to its competitive position, the state of the economy and global factors.

In weak-form markets, fundamentalists, who make investment decisions on the expectations of individual firms, should therefore be able to "out-guess" chartists and profit to the extent that such information is not assimilated into past prices (a phenomenon particularly applicable to companies whose financial securities are infrequently traded). However, if the semi-strong form is true, fundamentalists can no longer gain from their research.

The strong form declares that current prices fully reflect *all information*, which not only includes all publically available information but also *insider* knowledge. As a consequence, unless they are lucky, even the most privileged investors cannot profit in the long term from trading financial securities before their price changes. In the presence of strong form efficiency the market price of any financial security should represent its intrinsic (true) value based on anticipated returns and their degree of risk.

> So, as the EMH strengthens, speculative profit opportunities weaken. Competition among large numbers of increasingly well-informed market participants drives security prices to a consensus value, which reflects the best possible forecast of a company's uncertain future prospects.

Which strength of the EMH best describes the capital market and whether investors can ever "beat the market" need not concern us here. The point is that whatever levels of efficiency the market exhibits (weak, semi- strong or strong):

- Current prices reflect all the relevant information used by that market (price history, public data and insider information, respectively).
- Current prices only change when new information becomes available.

It follows, therefore that prices must follow a "random walk" to the extent that new information is *independent* of the last piece of information, which they have already absorbed.

- And it this phenomenon that has the most important consequences for how management model their strategic investment-financing decisions to maximise shareholder wealth

Activity 2
Before we continue, you might find it useful to review the Chapter so far and briefly summarise the main points..

1.3 The Role of Mean-Variance Efficiency

We began the Chapter with an idealised picture of investors (including management) who are rational and risk-averse and formally analyse one course of action in relation to another. What concerns them is not only profitability but also the likelihood of it arising; a *risk-return* trade-off with which they feel comfortable and that may also be unique.

Thus, in a sophisticated mixed market economy where ownership is divorced from control, it follows that the objective of strategic financial management should be to implement optimum investment-financing decisions using risk-adjusted wealth maximising criteria, which satisfy a multiplicity of shareholders (who may already hold a diverse portfolio of investments) by placing them all in an equal, optimum financial position.

No easy task!

But remember, we have not only assumed that investors are rational but that capital markets are also reasonably efficient at processing information. And this greatly simplifies matters for management. Because today's price is *independent* of yesterday's price, efficient markets have *no memory* and individual security price movements are *random*. Moreover, investors who comprise the market are so large in number that no one individual has a comparative advantage. In the short run, "you win some, you lose some" but long term, investment is a *fair game* for all, what is termed a "martingale". As a consequence, management can now afford to take a *linear* view of investor behaviour (as new information replaces old information) and model its own plans accordingly.

> What rational market participants require from companies is a diversified investment portfolio that delivers a maximum return at minimum risk.
>
> What management need to satisfy this objective are investment-financing strategies that maximise corporate wealth, validated by simple *linear* models that statistically quantify the market's risk-return *trade-off.*

Like Fisher's Separation Theorem, the concept of linearity offers management a lifeline because in *efficient* capital markets, rational investors (including management) can now assess anticipated investment returns (r_i) by reference to their probability of occurrence, (p_i) using classical statistical theory.

If the returns from investments are assumed to be *random*, it follows that their *expected return* (R) is the expected monetary value (EMV) of a symmetrical, *normal* distribution (the familiar "bell shaped curve" sketched overleaf). Risk is defined as the *variance* (or dispersion) of individual returns: the greater the variability, the greater the risk.

Unlike the mean, the statistical measure of dispersion used by the market or management to assess risk is partly a matter of convenience. The *variance* (VAR) or its square root, the *standard deviation* ($\sigma = \sqrt{VAR}$) is used.

When considering the *proportion* of risk due to some factor, the variance (VAR = σ^2) is sufficient. However, because the standard deviation (σ) of a normal distribution is measured in the same units as (R) the expected value (whereas the variance (σ^2) only summates the squared deviations around the mean) it is more convenient as an *absolute* measure of risk.

Moreover, the standard deviation (σ) possesses another attractive statistical property. Using confidence limits drawn from a Table of *z* statistics, it is possible to establish the *percentage probabilities* that a random variable lies within *one, two or three standard deviations above, below* or *around* its expected value, also illustrated below.

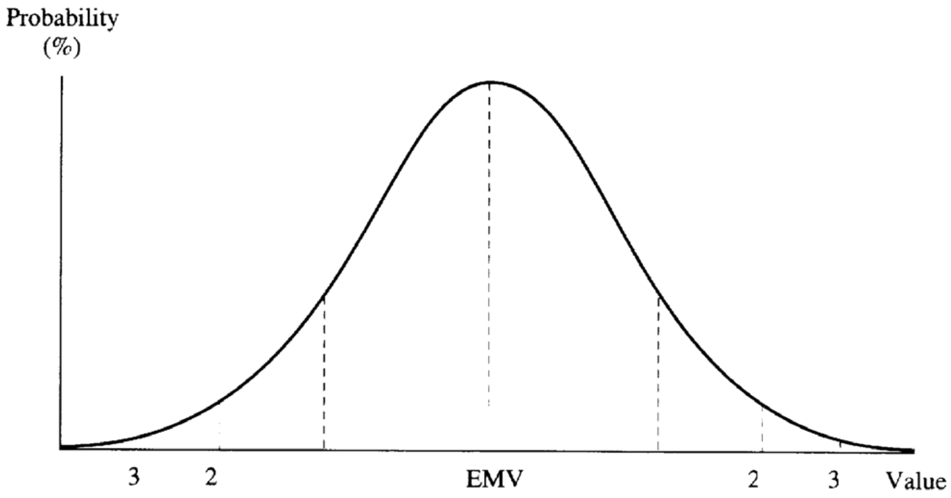

Figure 1.1: The Symmetrical Normal Distribution, Area under the Curveand Confidence Limits

Armed with this statistical information, investors and management can then accept or reject investments according to the degree of confidence they wish to attach to the likelihood (risk) of their desired returns. Using decision rules based upon their optimum criteria for *mean-variance efficiency,* this implies management and investors should pursue:

- Maximum expected return (R) for a given level of risk, (s).
- Minimum risk (s) for a given expected return (R).

Thus, our conclusion is that if modern capital market theory is based on the following three assumptions:

(i) Rational investors,
(ii) Efficient markets,
(iii) Random walks.

The normative wealth maximisation objective of strategic financial management requires the optimum selection of a portfolio of investment projects, which maximises their expected return (R) commensurate with a degree of risk (s) acceptable to existing shareholders and potential investors.

Activity 3

If you are not familiar with the application of classical statistical formulae to financial theory, read Chapter Four of "Strategic Financial Management" (both the text and exercises) downloadable from bookboon.com.

Each chapter focuses upon the two essential characteristics of investment, namely expected return and risk. The calculation of their corresponding statistical parameters, the mean of a distribution and its standard deviation (the square root of the variance) applied to investor utility should then be familiar.

We can then apply simple mathematical notation: (r_i, p_i, R, VAR, σ and U) to develop a more complex series of ideas throughout the remainder of this text.

1.4 The Background to Modern Portfolio Theory

From our preceding discussion, rational investors in reasonably efficient markets can assess the likely profitability of *individual* corporate investments by a statistical weighting of their expected returns, based on a *normal* distribution (the familiar bell-shaped curve).

- Rational-risk averse investors expect either a *maximum* return for a *given* level of risk, or a *given* return for *minimum* risk.
- Risk is measured by the standard deviation of returns and the overall expected return is measured by its weighted probabilistic average.

Using mean-variance efficiency criteria, investors then have *three* options when managing a *portfolio* of investments depending on the performance of its individual components.

(i) To trade (buy or sell),
(ii) To hold (do nothing),
(iii) To substitute (for example, shares for loan stock).

However, it is important to note that what any individual chooses to do with their portfolio constituents cannot be resolved by *statistical* analyses alone. Ultimately, their behaviour depends on how they interpret an investment's risk-return trade off, which is measured by their *utility curve*. This calibrates the individual's *current* perception of risk concerning uncertain *future* gains and losses. Theoretically, these curves are simple to calibrate, but less so in practice. Risk attitudes not only differ from one investor to another and may be unique but can also vary markedly over time. For the moment, suffice it to say that there is no *universally* correct decision to trade, hold, or substitute one constituent relative to another within a financial investment portfolio.

Review Activity

1. Having read the fourth chapters of the following series from bookboon.com recommended in Activity 3:
   Strategic Financial Management (SFM),
   Strategic Financial Management; Exercises (SFME).

- In *SFM*: pay particular attention to Section 4.5 onwards, which explains the relationship between *mean-variance* analyses, theconcept of *investor utility* and the application of *certainty equivalent* analysis to investment appraisal.
- In *SFME:* work through Exercise 4.1.

2. Next download the free companion text to this e-book:
   Portfolio Theory and Financial Analyses; Exercises (PTFAE), 2010.
3. Finally, read Chapter One of *PTFAE.*

It will test your understanding so far. The exercises and solutions are presented logically as a guide to further study and are easy to follow. Throughout the remainder of the book, each chapter's exercises and equations also follow the same structure of this text. So throughout, you should be able to complement and reinforce your theoretical knowledge of modern portfolio theory (MPT) at *your own* pace.

1.5 Summary and Conclusions

Based on our Review Activity, there are two interrelated questions that we have not yet answered concerning any wealth maximising investor's risk-return trade off, irrespective of their behavioural attitude towards risk.

> What if investors don't want "to put all their eggs in one basket" and wish to diversify beyond a *single* asset portfolio?
>
> How do financial management, acting on their behalf, incorporate the *relative* risk-return trade-off between a *prospective* project and the firm's *existing* asset portfolio into a quantitative model that still maximises wealth?

To answer these questions, throughout the remainder of this text and its exercise book, we shall analyse the evolution of Modern Portfolio Theory (MPT).

Statistical calculations for the expected risk-return profile of a *two-asset* investment portfolio will be explained. Based upon the mean-variance efficiency criteria of Harry Markowitz (1952) we shall begin with:

- The risk-reducing effects of a diverse two-asset portfolio,
- The optimum two-asset portfolio that minimises risk, with individual returns that are perfectly (negatively) correlated.

We shall then extend our analysis to *multi-asset* portfolio optimisation, where John Tobin (1958) developed the *capital market line* (CML) to show how the introduction of risk-free investments define a "frontier" of efficient portfolios, which further reduces risk. We discover, however, that as the size of a portfolio's constituents increase, the mathematical calculation of the variance is soon dominated by covariance terms, which makes its computation unwieldy.

Fortunately, the problem is not insoluble. Ingenious, subsequent developments, such as the *specific* capital asset pricing model (CAPM) formulated by Sharpe (1963) Lintner (1965) and Mossin (1966), the option-pricing model of Black and Scholes (1973) and *general* arbitrage pricing theory (APT) developed by Ross (1976), all circumvent the statistical problems encountered by Markowitz.

By dividing *total* risk between *diversifiable* (unsystematic) risk and *undiversifiable* (systematic or market) risk, what is now termed Modern Portfolio Theory (MPT) explains how rational, risk averse investors and companies can price securities, or projects, as a basis for profitable portfolio trading and investment decisions. For example, a profitable trade is accomplished by buying (selling) an undervalued (overvalued) security relative to an appropriate stock market index of *systematic* risk (say the FT-SE All Share).This is measured by the *beta* factor of the individual security relative to the market portfolio. As we shall also discover it is possible for companies to define project betas for project appraisal that measure the systematic risk of specific projects.

So, there is much ground to cover. Meanwhile, you should find the diagram in the Appendix provides a useful road-map for your future studies.

1.6 Selected References

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Part II:
The Portfolio Decision

2 Risk and Portfolio Analysis

Introduction

We have observed that *mean-variance efficiency* analyses, premised on investor rationality (maximum return) and risk aversion (minimum variability), are not always sufficient criteria for investment appraisal. Even if investments are considered in isolation, wealth maximising accept-reject decisions depend upon an individual's perception of the riskiness of its expected future returns, measured by their personal *utility curve*, which may be unique.

Your reading of the following material from the *bookboon.com* companion texts, recommended for Activity 3 and the Review Activity in the previous chapter, confirms this.

- *Strategic Financial Management* (*SFM*): Chapter Four, Section 4.5 onwards,
- *SFM*; *Exercises (SFME)*: Chapter Four, Exercise 4.1,
- *SFM*: *Portfolio Theory and Analyses; Exercises (PTAE)*: Chapter One.

Any conflict between mean-variance efficiency and the concept of investor utility can only be resolved through the application of *certainty equivalent* analysis to investment appraisal. The ultimate test of *statistical* mean-variance analysis depends upon *behavioural* risk attitudes.

So far, so good, but there is now another complex question to answer in relation to the search for future wealth maximising investment opportunities:

> Even if there is only *one* new investment on the horizon, including a *choice* that is either *mutually exclusive,* or if *capital is rationed*, (i.e. the acceptance of one precludes the acceptance of others).
>
> How do individuals, or companies and financial institutions that make decisions on their behalf, incorporate the *relative* risk-return trade-off between a *prospective* investment and an *existing* asset portfolio into a quantitative model that still maximises wealth?

2.1 Mean-Variance Analyses: Markowitz Efficiency

Way back in 1952 without the aid of computer technology, H.M. Markowitz explained why rational investors who seek an *efficient* portfolio (one which minimises risk without impairing return, or maximises return for a given level of risk) by introducing new (or off-loading existing) investments, cannot rely on mean-variance criteria alone.

> Even before *behavioural* attitudes are calibrated, Harry Markowitz identified a *third statistical* characteristic concerning the risk-return relationship between individual investments (or in management's case, capital projects) which justifies their inclusion within an *existing* asset portfolio to maximise wealth.

To understand Markowitz' train of thought; let us begin by illustrating his simple *two asset case*, namely the construction of an *optimum* portfolio that comprises two investments. Mathematically, we shall define their expected returns as $R_i(A)$ and $R_i(B)$ respectively, because their size depends upon which one of two future economic "states of the world" occur. These we shall define as S_1 and S_2 with an equal probability of occurrence. If S_1 prevails, $R_1(A) > R_1(B)$. Conversely, given S_2, then $R_2(A) < R_2(B)$. The numerical data is summarised as follows:

Return\State	S_1	S_2
$R_i(A)$	20%	10%
$R_i(B)$	10%	20%

> **Activity 1**
>
> The overall expected return R(A) for investment A (its mean value) is obviously 15 per cent (the weighted average of its expected returns, where the weights are the probability of each state of the world occurring. Its risk (range of possible outcomes) is between 10 to 20 per cent. The same values also apply to B.
>
> Mean-variance analysis therefore informs us that because R(A) = R(B) and σ (A) = σ (B), we should all be *indifferent* to either investment. Depending on your behavioural attitude towards risk, one is perceived to be as good (or bad) as the other. So, either it doesn't matter which one you accept, or alternatively you would reject both.
>
> - Perhaps you can confirm this from your reading for earlier Activities?

However, the question Markowitz posed is whether there is an *alternative* strategy to the exclusive selection of either investment or their wholesale rejection? And because their respective returns do not move in *unison* (when one is good, the other is bad, depending on the state of the world) his answer was yes.

By not "putting all your eggs in one basket", there is a *third* option that in our example produces an *optimum* portfolio *i.e.* one with the *same* overall return as its constituents but with *zero* risk.

If we *diversify* investment and *combine* A and B in a *portfolio* (P) with half our funds in each, then the overall portfolio return R(P) = 0.5R(A) + 0.5R(B) still equals the 15 per cent mean return for A and B, whichever state of the world materialises. Statistically, however, our new portfolio not only has the same return, R(P) = R(A) = R(B) but the risk of its constituents, σ(A) = σ(B), is also eliminated entirely. Portfolio risk; σ(P) = 0. Perhaps you can confirm this?

Activity 2

As we shall discover, the previous example illustrates an *ideal* portfolio scenario, based upon your entire knowledge of investment appraisal under conditions of risk and uncertainty explained in the *SFM* texts referred to earlier. So, let us summarise their main points

- An *uncertain* investment is one with a *plurality* of cash flows whose probabilities are *non-quantifiable.*
- A *risky* investment is one with a *plurality* of cash flows to which we attach *subjective probabilities.*
- Expected returns are assumed to be characterised by a normal distribution (i.e. they are random variables).
- The probability density function of returns is defined by the mean-variance of their distribution.
- An efficient choice between individual investments maximises the discounted return of their anticipated cash flows and minimises the standard deviation of the return.

So, without recourse to further statistical analysis, (more of which later) but using your knowledge of investment appraisal:

Can you define the objective of portfolio theory and using our previous numerical example, briefly explain what Markowitz adds to our understanding of mean-variance analyses through the efficient diversification of investments?

For a given overall return, the objective of efficient portfolio diversification is to determine an overall standard deviation (level of risk) that is lower than any of its individual portfolio constituents.

According to Markowitz, three significant points arise from our simple illustration with one important conclusion that we shall develop throughout the text.

1) We can combine risky investments into a less risky, even risk-free, portfolio by "not putting all our eggs in one basket"; a policy that Markowitz termed *efficient* diversification, and subsequent theorists and analysts now term *Markowitz efficiency* (praise indeed).
2) A portfolio of investments may be preferred to all or some of its constituents, irrespective of investor risk attitudes. In our previous example, no rational investor would hold either investment exclusively, because diversification can maintain the *same* return for *less* risk.
3) Analysed in isolation, the risk-return profiles of individual investments are insufficient criteria by which to assess their true value. Returning to our example, A and B initially seem to be equally valued. Yet, an investor with a substantial holding in A would find that moving funds into B is an attractive proposition (and *vice versa*) because of the *inverse* relationship between the *timing* of their respective risk-return profiles, defined by likely states of the world. When one is good, the other is bad and *vice versa*.

> According to Markowitz, risk may be *minimised,* if not *eliminated* entirely without compromising overall return through the diversification and selection of an *optimum* combination of investments, which defines an *efficient* asset portfolio.

2.2 The Combined Risk of Two Investments

So, in general terms, how do we derive (model) an optimum, efficient diversified portfolio of investments?

To begin with, let us develop the "two asset case" where a company have funds to invest in two profitable projects, A and B. One proportion x is invested in A and $(1-x)$ is invested in B.

We know from Activity 1 that the *expected return from a portfolio* R(P) is simply a weighted average of the expected returns from two projects, R(A) and R(B), where the weights are the proportional funds invested in each. Mathematically, this is given by:

(1) $R(P) = x\,R(A) + (1 - x)\,R(B)$

But, what about the likelihood (probability) of the portfolio return R(P) occurring?

Markowitz defines the *proportionate* risk of a two-asset investment as the *portfolio variance:*

(2) $VAR(P) = x^2\,VAR(A) + (1-x)^2\,VAR(B) + 2x(1-x)\,COV(A, B)$

Percentage risk is then measured by the *portfolio standard deviation* (i.e. the square root of the variance):

(3) $\sigma(P) = \sqrt{VAR(P)} = \sqrt{[\,x^2\,VAR(A) + (1-x)^2\,VAR(B) + 2x(1-x)\,COV(A, B)]}$

Unlike the risk of a *single* random variable, the variance (or standard deviation) of a *two-asset* portfolio exhibits *three* separable characteristics:

1) The risk of the constituent investments measured by their respective variances,
2) The squared proportion of available funds invested in each,
3) The relationship between the constituents measured by twice the *covariance*.

The *covariance* represents the variability of the combined returns of individual investments around their mean. So, if A and B represent two investments, the degree to which their returns ($r_i A$ and $r_i B$) vary together is defined as:

(4) $$COV(A,B) = \sum_{i=1}^{n} \{ [(r_i A - R(A))] [(r_i B - R(B))]\, p_i \}$$

For each observation i, we multiply three terms together: the deviation of $r_i(A)$ from its mean R(A), the deviation of $r_i(B)$ from its mean R(B) and the probability of occurrence p_i. We then add the results for each observation.

Returning to Equations (2) and (3), the covariance enters into our portfolio risk calculation *twice* and is *weighted* because the *proportional* returns on A vary with B and *vice versa*.

Depending on the state of the world, the logic of the covariance itself is equally simple.

- If the returns from two investments are *independent* there is no observable relationship between the variables and knowledge of one is of no use for predicting the other. The variance of the two investments combined will equal the sum of the individual variances, *i.e.* the covariance is *zero*.
- If returns are *dependent* a relationship exists between the two and the covariance can take on either a positive or negative value that affects portfolio risk.

1) When each paired deviation around the mean is negative, their product is positive and so too, is the covariance.
2) When each paired deviation is positive, the covariance is still positive.
3) When one of the paired deviations is negative their covariance is negative.

Thus, in a state of the world where individual returns are *independent* and whatever happens to one affects the other to opposite effect, we can reduce risk by diversification without impairing overall return.

Under condition (iii) the portfolio variance will obviously be less than the sum of its constituent variances. Less obvious, is that when returns are *dependent*, risk reduction is still possible.

To demonstrate the application of the statistical formulae for a two-asset portfolio let us consider an equal investment in two corporate capital projects (A and B) with an equal probability of producing the following paired cash returns.

Pi	A	B
	%	%
0.5	8	14
0.5	12	6

We already know that the expected return on each investment is calculated as follows:

R(A) = (0.5 × 8) + (0.5 × 12) = 10%

R(B) = (0.5 × 14) + (0.5 × 6) = 10%

Using Equation (1), the *portfolio return* is then given by:

R(P) = (0.5 × 10) + (0.5 × 10) = 10%

Since the portfolio return equals the expected returns of its constituents, the question management must now ask is whether the decision to place funds in both projects in equal proportions, rather than A or B exclusively, reduces risk?

To answer this question, let us first calculate the variance of A, then the variance of B and finally, the covariance of A and B. The data is summarised in Table 2.1 below.

With a negative covariance value of minus 8, combining the projects in equal proportions can obviously reduce risk. The question is by how much?

Probability	Deviations		VAR(A)	VAR(B)	COV(A,B)
	A	B			
p_i	$(r_i - R)$	$(r_i - R)$	$(r_i - R)^2 p_i$	$(r_i - R)^2 p_i$	$[r_i A - R(A)][(r_i B - R(B)]p_i$
0.5	(2)	4	2	8	(4)
0.5	2	(4)	2	8	(4)
1.0	0	0	4	16	(8)

Table 2.1: The Variances of Two Investments and their Covariance

Using Equation (2), let us now calculate the portfolio variance:

VAR(P) = (0.5^2 × 4) + (0.5^2 × 16) + (2 × 0.5) (0.5 × -8) = 1

And finally, the *percentage* risk given by Equation (3), the portfolio standard deviation:

σ(P) = √ VAR(P) = √ 1 = 1%

Activity 3

Unlike our original example, which underpinned Activities 1 and 2, the current statistics reveal that this portfolio is not *riskless* (i.e. the percentage risk represented by the standard deviation σ *is not zero*). But given that our investment criteria remain the same (either *minimise* σ, given R; or *maximise* R *given* σ) the next question to consider is how the portfolio's risk-return profile compares with those for the individual projects. In other words is diversification beneficial to the company?

If we compare the standard deviations for the portfolio, investment A and investment B with their respective expected returns, the following relationships emerge.

$$\sigma(P) < \sigma(A) < \sigma(B);\ \textit{given}\ R(P) = R(A) = R(B)$$

These confirm that our decision to place funds in both projects in equal proportions, rather than either A or B exclusively, is the correct one. You can verify this by deriving the standard deviations for the portfolio and each project from the variances in the Table 2.1.

Investments	R %	σ %
P (0.5A + 0.5B)	10	1.00
A	10	2.00
B	10	4.00

2.3 The Correlation between Two Investments

Because the covariance is an *absolute* measure of the correspondence between the movements of two random variables, its interpretation is often difficult. Not all paired deviations need be negative for diversification to produce a degree of risk reduction. If we have small or large negative or positive values for individual pairs, the covariance may also assume small or large values either way. So, in our previous example, COV(A, B) = minus 8. But what does this mean exactly?

Fortunately, we need not answer this question? According to Markowitz, the statistic for the *linear correlation coefficient* can be substituted into the third covariance term of our equation for portfolio risk to simplify its interpretation. With regard to the mathematics, beginning with the variance for a two asset portfolio:

(2) $VAR(P) = x^2 \, VAR(A) + (1\text{-}x)^2 \, VAR(B) + 2x \, (1\text{-}x) \, COV(A,B)$

Let us define the correlation coefficient.

(5) $COR(A,B) = \dfrac{COV(A,B)}{\sigma A \, \sigma B}$

Now rearrange terms to redefine the covariance.

(6) $COV(A,B) = COR(A,B) \, \sigma A \, \sigma B$

Clearly, the portfolio variance can now be measured by the substitution of Equation (6) for the covariance term in Equation (2).

(7) $VAR(P) = x^2 \, VAR(A) + (1\text{-}x)^2 \, VAR(B) + 2x \, (1\text{-}x) \, COR(A,B) \, \sigma A \, \sigma B$

The standard deviation of the portfolio then equals the square root of Equation (7):

(8) $\sigma (P) = \sqrt{} \, VAR (P) = \sqrt{} \, [x^2 \, VAR(A) + (1\text{-}x)^2 \, VAR (B) + 2x(1\text{-}x) \, COR(A,B) \, \sigma A \, \sigma B]$

Activity 4

So far, so good; we have proved mathematically that the correlation coefficient can replace the covariance in the equations for portfolio risk.

But, given your knowledge of statistics, can you now explain why Markowitz thought this was a significant contribution to portfolio analysis?

Like the standard deviation, the correlation coefficient is a *relative* measure of variability with a convenient property. Unlike the covariance, which is an *absolute* measure, it has only *limited values between +1 and -1*. This arises because the coefficient is calculated by taking the covariance of returns and dividing by the product (multiplication) of the individual standard deviations that comprise the portfolio. Which is why, for two investments (A and B) we have defined:

$$(5)\quad COR(A,B) = \frac{COV(A,B)}{\sigma A\ \sigma B}$$

The correlation coefficient therefore measures the extent to which two investments vary together as a *proportion* of their respective standard deviations. So, if two investments are *perfectly* and *linearly* related, they deviate by *constant proportionality*.

Of course, the interpretation of the correlation coefficient still conforms to the logic behind the covariance, but with the advantage of limited values.

- If returns are *independent*, i.e. no relationship exists between two variables; their correlation will be zero (although, as we shall discover later, risk can still be reduced by diversification).
- If returns are *dependent*:

1) A perfect, positive correlation of +1 means that whatever affects one variable will equally affect the other. Diversified risk-reduction is *not possible.*
2) A perfect negative correlation of -1 means that an *efficient* portfolio can be constructed, with *zero* variance exhibiting *minimum* risk. One investment will produce a return above its expected return; the other will produce an equivalent return below its expected value and *vice versa*.
3) Between +1 and –1, the correlation coefficient is determined by the proximity of direct and inverse relationships between individual returns So, in terms of risk reduction, even a low positive correlation can be beneficial to investors, depending on the allocation of total funds at their disposal.

Providing the correlation coefficient between returns is less than +1, all investors (including management) can profitably diversify their portfolio of investments. Without compromising the overall return, relative portfolio risk measured by the standard deviation will be less than the weighted average standard deviation of the portfolio's constituents.

Review Activity

Using the statistics generated by Activity 3, confirm that the substitution of the correlation coefficient for the covariance into our revised equations for the portfolio variance and standard deviation does not change their values, or our original investment decision?

Let us begin with a summary of the previous mean-variance data for the two-asset portfolio:

R(P) = 0.5 R(A) + 0.5 R(B)	VAR(P)	VAR(A)	VAR(B)	COV(A,B)
10%	1	4	16	(8)

The correlation coefficient is given by:

$$(5)\quad COR(A,B) = \frac{COV(A,B)}{\sigma A \;\; \sigma B} = \frac{-8}{\sqrt{4}.\sqrt{16}} = -1$$

Substituting this value into our revised equations for the portfolio variance and standard deviation respectively, we can now confirm our initial calculations for Activity 3.

(7) VAR(P) = x^2 VAR(A) + $(1-x)^2$ VAR(B) + 2x (1-x)COR(A,B) σ A σ B
= ($0.5^2 \times 4$) + ($0.5^2 \times 16$) + {2 × 0.5(1-0.5) × -1(2 × 4)}
= 1

(8) σ (P) = √ VAR(P) = √ 1.0
= 1.00 %

Thus, the company's original *portfolio* decision to place an equal proportions of funds in both investments, rather than either A or B *exclusively*, still applies. This is also confirmed by a summary of the following inter-relationships between the risk-return profiles of the portfolio and its constituents, which are identical to our previous Activity.

σ(P) = 1.00% < σ(A) = 2.00 % < σ(B) = 4.00%; *given* R(P) = R(A) = R(B) = 10%

2.4 Summary and Conclusions

It should be clear from our previous analyses that the risk of a *two-asset* portfolio is a function of its covariability of returns. Risk is at a *maximum* when the correlation coefficient between two investments is +1 and at a *minimum* when the correlation coefficient equals -1. For the vast majority of cases where the correlation coefficient is between the two, it also follows that there will be a *proportionate* reduction in risk, relative to return. Overall portfolio risk will be less than the weighted average risks of its constituents. So, investors can still profit by diversification because:

σ(P) > σ(P) > σ(P), given R(P), if COR(A,B) = +1, 0, or -1, respectively.

2.5 Selected References

1. Hill, R.A., bookboon.com
   - *Strategic Financial Management,* 2009.
   - *Strategic Financial Management; Exercises,* 2009.
   - *Portfolio Theory and Financial Analyses; Exercises,* 2010.
2. Markowitz, H.M., "Portfolio Selection", *The Journal of Finance,* Vol. 13, No. 1, March 1952.

3 The Optimum Portfolio

Introduction

In an efficient capital market where the random returns from two investments are normally distributed (symmetrical) we have explained how rational (risk averse) investors and companies who seek an optimal portfolio can maximise their utility preferences by *efficient* diversification. Any combination of investments produces a trade-off between the two statistical parameters that define a normal distribution; their expected return and standard deviation (risk) associated with the *covariability* of individual returns. According to Markowitz (1952) this is best measured by the *correlation coefficient* such that:

> *Efficient* diversified portfolios are those which *maximise* return for a *given* level of risk, or *minimise* risk for a *given* level of return for different correlation coefficients.

The purpose of this chapter is to prove that when the correlation coefficient is at a minimum and portfolio risk is minimised we can derive an *optimum portfolio* of investments that maximises there overall expected return.

3.1 The Mathematics of Portfolio Risk

You recall from Chapter Two (both the Theory and Exercises texts) that substituting the *relative* linear correlation coefficient for the *absolute* covariance term into a two-asset portfolio's standard deviation simplifies the wealth maximisation analysis of the risk-return trade-off between the covariability of returns. Whenever the coefficient falls below one, there will be a *proportionate* reduction in portfolio risk, relative to return, by diversifying investment.

For example, given the familiar equations for the return, variance, correlation coefficient and standard deviation of a two-asset portfolio:

(1) $R(P) = x\, R(A) + (1-x)\, R(B)$

(2) $VAR(P) = x^2\, VAR(A) + (1-x)^2\, VAR(B) + 2x\,(1-x)\, COV(A,B)$

(5) $COR(A,B) = \frac{COV(A,B)}{\sigma A\, \sigma B}$

(8) $\sigma(P) = \sqrt{VAR(P)} = \sqrt{[\, x^2\, VAR(A) + (1-x)^2\, VAR(B) + 2x(1-x)\, COR(A,B)\, \sigma A\, \sigma B]}$

Harry Markowitz (*op. cit.*) proved mathematically that:

$$\sigma(P) > \sigma(P) > \sigma(P), \text{ given } R(P), \text{ if } COR(A,B) = +1, 0, \text{ or } -1, \text{ respectively.}$$

However, he also illustrated that if the returns from two investments exhibit *perfect positive, zero*, or *perfect negative* correlation, then portfolio risk measured by the standard deviation using Equation (8) can be simplified further.

To understand why, let us return to the original term for the portfolio variance:

(2) $VAR(P) = x^2\, VAR(A) + (1-x)^2\, VAR(B) + 2x\,(1-x)\, COV(A,B)$

Because the correlation coefficient is given by:

(5) $COR(A,B) = \dfrac{COV(A,B)}{\sigma A\, \sigma B}$

We can rearrange its terms, just as we did in Chapter Two, to redefine the covariance:

(6) COV(A,B) = COR(A,B) σ A σ B

The portfolio variance can now be measured by the substitution of Equation (6) for the covariance term in Equation (2), so that.

(7) VAR(P) = x^2 VAR(A) + $(1-x)^2$ VAR(B) + 2x (1-x) COR(A,B) σ A σ B

The standard deviation of the portfolio then equals the square root of Equation (7):

(8) σ (P) = √ VAR(P) = √ [x^2 VAR(A) + $(1-x)^2$ VAR(B) + 2x(1-x) COR(A,B) σ A σ B]

Armed with this information, we can now confirm that:

> If the returns from two investments exhibit perfect, positive correlation, portfolio risk is simply the weighted average of its constituent's risks and at a maximum.
>
> σ(P) = x σ(A) + (1-x) σ(B)

If the correlation coefficient for two investments is positive and COR(A,B) also equals plus one, then the correlation term can disappear from the portfolio risk equations without affecting their values. The portfolio variance can be rewritten as follows:

(9) VAR(P) = x^2 VAR(A) + $(1 - x)^2$ VAR(B) + 2x (1-x) σ(A) σ(B)

Simplifying, this is equivalent to:

(10) VAR(P) = $[x\, \sigma(A) + (1-x)\, \sigma(B)]^2$

And because this is a *perfect square*, our probabilistic estimate for the risk of a two-asset portfolio measured by the standard deviation given by Equation (8) is equivalent to:

(11) σ(P) = √ VAR(P) = x σ(A) + (1-x)σ(B)

To summarise:

> Whenever COR(A, B) = +1 (perfect positive) the portfolio variance VAR(P) and its square root, the standard deviation σ(P), simplify to the weighted average of the respective statistics, based on the probabilistic returns for the individual investments.

But this is not all. The substitution of Equation (6) into the expression for portfolio variance has two further convenient properties. Given:

(6) COV(A,B) = COR(A,B) σA σB

If the relationship between two investments is *independent* and exhibits *zero* correlation, the portfolio variance given by Equation (7) simplifies to:

(12) VAR(P) = x^2 VAR(A) + $(1-x)^2$ VAR(B)

And its corresponding standard deviation also simplifies:

(13) σ(P) = √ [x^2 VAR(A) + $(1-x)^2$ VAR(B)]

Similarly, with *perfect inverse* correlation we can deconstruct our basic equations to simplify the algebra.

Activity 1

When the correlation coefficient for two investments is perfect positive and equals one, the correlation term disappears from equations for portfolio risk without affecting their values. The portfolio variance VAR(P)) and its square root, the standard deviation σ(P), simplify to the *weighted average* of the respective statistics.

Can you manipulate the previous equations to prove that if COR(A,B) equals *minus* one (perfect negative) they still correspond to a weighted average, like their perfect positive counterpart, but with one fundamental difference? Whenever COR(A, B) = +1 (perfect positive) the portfolio variance VAR(P) and its square root, the standard deviation σ(P), simplify to the weighted average of the respective statistics, based on the probabilistic returns for the individual investments.

Let us begin again with the familiar equation for portfolio variance.

(7) VAR(P) = x^2 VAR(A) + $(1-x)^2$ VAR(B) + $2x$ $(1-x)$ COR(A,B) σ A σ B

If the correlation coefficient for two investments is negative and COR(A,B) also equals minus one, then the coefficient can disappear from the equation's third right hand term without affecting its value. It can be rewritten as follows with only a change of sign (positive to negative):

(14) VAR(P) = x^2 VAR(A) + $(1-x)^2$ VAR(B) – $2x$ $(1-x)$ σ(A) σ(B)

Simplifying, this is equivalent to:

(15) VAR(P) = $[x\, σ(A) – (1-x)\, σ(B)]^2$

And because this is a *perfect square*, our probabilistic estimate for the risk of a two-asset portfolio measured by the standard deviation is equivalent to:

(16) $\sigma(P) = x\,\sigma(A) - (1\text{-}x)\,\sigma(B)$

> The only difference between the formulae for the risk of a two-asset portfolio where the correlation coefficient is at either limit (+1 or –1) is simply a matter of sign (positive or negative) in the right hand term for σ (P).

3.2 Risk Minimisation and the Two-Asset Portfolio

When investment returns exhibit perfect positive correlation a portfolio's risk is at a maximum, defined by the weighted average of its constituents. As the correlation coefficient falls there is a proportionate reduction in portfolio risk relative to this weighted average. So, if we diversify investments; risk is minimised when the correlation coefficient is minus one.

> To illustrate this general proposition, Figure 3.1 roughly sketches the various two-asset portfolios that are possible if corporate management combine two investments, A and B, in various proportions for different correlation coefficients.

Specifically, the *diagonal* line A (+1) B; the *curve* A (E) B and the *"dog-leg"* A (-1) B are the focus of all possible risk-return combinations when our correlation coefficients equal plus one, zero and minus one, respectively.

Thus, if project returns are perfectly, positively correlated we can construct a portfolio with any risk-return profile that lies along the *horizontal* line, A (+1) B, by varying the proportion of funds placed in each project. Investing 100 percent in A produces a minimum return but minimises risk. If management put all their funds in B, the reverse holds. Between the two extremes, having decided to place say two-thirds of funds in Project A, and the balance in Project B, we find that the portfolio lies one third along A (+1) B at point +1.

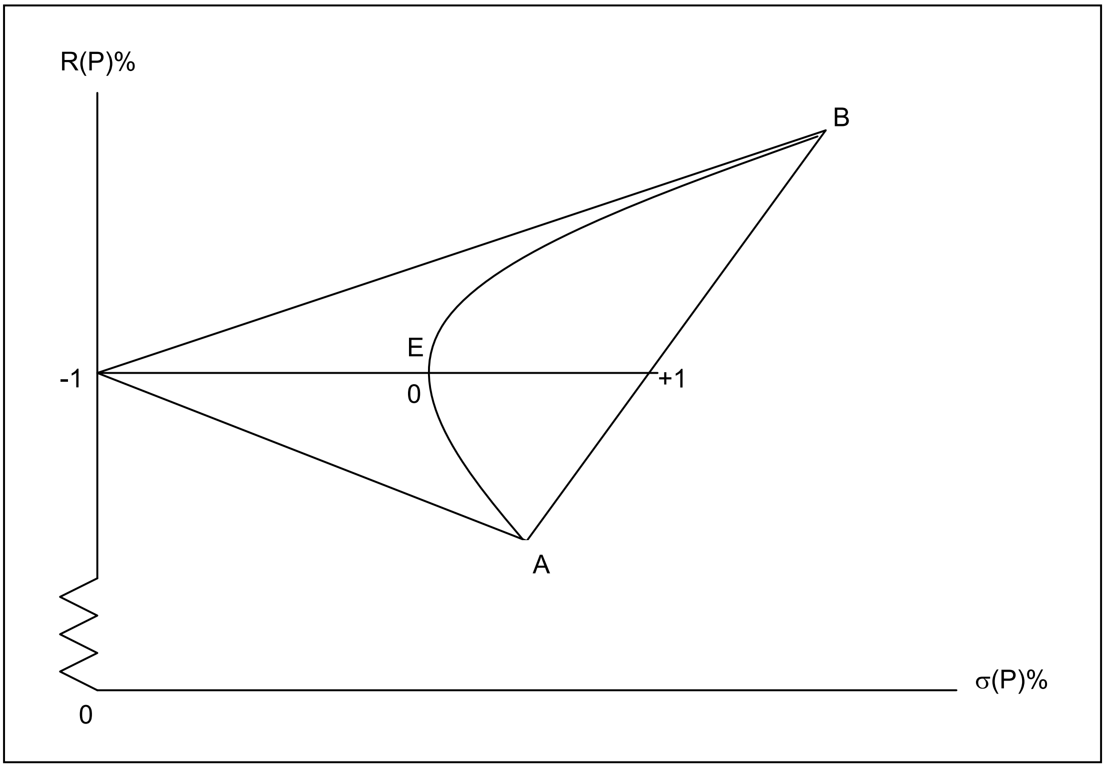

Figure 3.1: The Two Asset Risk-Return Profile and the Correlation Coefficient

Similarly, if the two returns exhibit perfect negative correlation, we could construct any portfolio that lies along the line A (-1) B. However, because the correlation coefficient equals minus one, the line is no longer straight but a *dog-leg* that also touches the vertical axis where σ(P) equals zero. As a consequence, our choice now differs on two counts.

- It is possible to construct a *risk-free* portfolio.
- No rational, risk averse investor would be interested in those portfolios which offer a *lower* expected return for the *same* risk.

As you can observe from Figure 3.1, the investment proportions lying along the line -1 to B offer higher returns for a given level of risk relative to those lying between -1 and A. Using the terminology of Markowitz based on mean-variance criteria; the first portfolio set is *efficient* and acceptable whilst the second is *inefficient* and irrelevant. The line -1 to B, therefore, defines the *efficiency frontier* for a two-asset portfolio.

Where the two lines meet on the vertical axis (point -1 on our diagram) the portfolio standard deviation is zero. As the *horizontal* line (-1, 0, +1) indicates, this *riskless* portfolio also conforms to our decision to place two-thirds of funds in Project A and one third in Project B.

Finally, in most cases where the correlation coefficient lies somewhere between its extreme value, every possible two-asset combination always lies along a *curve*. Figure 3.1 illustrates the risk-return trade-off assuming that the portfolio correlation coefficient is *zero*. Once again, because the data set is not perfect positive (less than +1) it turns back on itself. So, only a proportion of portfolios are efficient; namely those lying along the E-B frontier. The remainder, E-A, is of no interest whatsoever. You should also note that whilst risk is not eliminated entirely, it could still be *minimised* by constructing the appropriate portfolio, namely point E on our curve.

3.3 The Minimum Variance of a Two-Asset Portfolio

Investors trade financial securities to earn a return in the form of dividends and capital gains. Companies invest in projects to generate net cash inflows on behalf of their shareholders. Returns might be higher or lower than anticipated and their variability is the cause of investment risk. Investors and companies can reduce risk by diversifying their portfolio of investments. The preceding analysis explains why risk minimisation represents an *objective* standard against which investors and management compare their variance of returns as they move from one portfolio to another.

To prove this proposition, you will have observed from Figure 3.1 that the decision to place two-thirds of our funds in Project A and one-third in Project B falls between E and A when COR(A,B) = 0. This is defined by point 0 along the horizontal line (-1, 0, +1).

Because portfolio risk is minimised at point E, with a higher return above and to the left in our diagram, the decision is clearly *suboptimal.* At one extreme, speculative investors or companies would place all their money in Project B at point B hoping to maximise their return (completely oblivious to risk). At the other, the most risk-averse among them would seek out the proportionate investment in A and B which corresponds to E. Between the two, a higher expected return could also be achieved for any degree of risk given by the curve E-A. Thus, all investors would move up to the efficiency frontier E-B and depending upon their risk attitudes choose an appropriate combination of investments above and to the right of E.

However, without a graph, let alone data to fall back on, this raises another fundamental question.

How do investors and companies mathematically model an optimum portfolio with minimum variance from first principles?

According to Markowitz (*op. cit*) the mathematical derivation of a two-asset portfolio with *minimum* risk is quite straightforward.

Where a proportion of funds x is invested in Project A and $(1-x)$ in Project B, the portfolio variance can be defined by the familiar equation:

$$(7)\quad \text{VAR(P)} = x^2\ \text{VAR(A)} + (1-x)^2\ \text{VAR(B)} + 2x\,(1-x)\ \text{COR(A,B)}\ \sigma\,\text{A}\ \sigma\,\text{B}$$

The value of x, for which Equation (7) is at a *minimum*, is given by *differentiating* VAR(P) with respect to x and setting $\Delta\text{VAR(P)} / \Delta x = 0$, such that:

$$(17)\quad x = \frac{\text{VAR(B)} - \text{COR(A,B)}\ \sigma(\text{A})\ \sigma(\text{B})}{\text{VAR(A)} + \text{VAR(B)} - 2\ \text{COR(A,B)}\ \sigma(\text{A})\ \sigma(\text{B})}$$

Since all the variables in the equation for minimum variance are now known, the risk-return trade-off can be solved. Moreover, if the correlation coefficient equals *minus one*, risky investments can be combined into a *riskless* portfolio by solving the following equation when the standard deviation is *zero*.

(18) $\sigma (P) = \sqrt{[x^2 \, VAR(A) + (1-x)^2 \, VAR(B) + 2x(1-x) \, COR(A,B) \, \sigma A \, \sigma B]} = \underline{0}$

Because this is a *quadratic* in one unknown (x) it also follows that to *eliminate* portfolio risk when COR(A,B) = -1, the proportion of funds (x) invested in Project A should be:

(19) $x = 1 - \dfrac{\sigma(A)}{\sigma(A) + \sigma(B)}$

Activity 2

Algebraically, mathematically and statistically, we have covered a lot of ground since Chapter Two. So, the previous section, like those before it, is illustrated by the numerical application of data to theory in the bookboon companion text.

Portfolio Theory and Financial Analyses; Exercises (PTFAE):
Chapter Three, 2010.

You might find it useful at this point in our analysis to cross-reference the appropriate Exercises (3.1 and 3.2) before we continue?

3.4 The Multi-Asset Portfolio

In efficient capital markets where the returns from *two* investments are normally distributed (symmetrical) we have explained how rational (risk averse) investors and companies who require an optimal portfolio can maximise their utility preferences by diversification. Any combination of investments produce a trade-off between the statistical parameters that define a normal distribution; the expected return and standard deviation (risk) associated with the covariance of individual returns.

Efficient diversified portfolios are those which *maximise* return for a *given* level of risk, or *minimise* risk for a *given* level of return for different correlation coefficients.

However, most investors, or companies and financial managers (whether they control capital projects or financial services (such as insurance premiums, pension funds or investment trusts) may be responsible for numerous investments. It is important, therefore, that we extend our analysis to portfolios with more than two constituents.

Theoretically, this is not a problem. According to Markowitz (*op cit.*) if individual returns, standard deviations and the covariance for each pair of returns are known, the portfolio return R(P), portfolio variance VAR(P) and a probabilistic estimate of portfolio risk measured by the standard deviation s(P), can be calculated.

For a *multi-asset* portfolio where the number of assets equals n and x_i represents the proportion of funds invested in each, such that:

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0$$

We can define the portfolio return and variance as follows

$$(20) \quad R(P) = \sum_{i=1}^{n} x_i R_i$$

$$(21) \quad VAR(P) = \sum_{i=1}^{n} x_i^2 VAR_i + \sum_{i \neq j}^{n}\sum x_i x_j COVij$$

The covariance term, COVij determines the degree to which variations in the return to one investment, i, can serve to offset the variability of another, j. The standard deviation is then derived in the usual manner.

$$(22) \quad \sigma(P) = \sqrt{VAR(P)}$$

Assuming we now wish to *minimise* portfolio risk for any given portfolio return; our financial objective is equally straightforward:

(23) MIN: $\sigma(P)$, Given $R(P) = K$ (constant)

This mathematical function combines Equation (22) which is to be *minimised*, with a constraint obtained by setting Equation (20) for the portfolio return equal to a *constant* (K):

Figure 3.2 illustrates all the different risk-return combinations that are available from a hypothetical multi investment scenario.

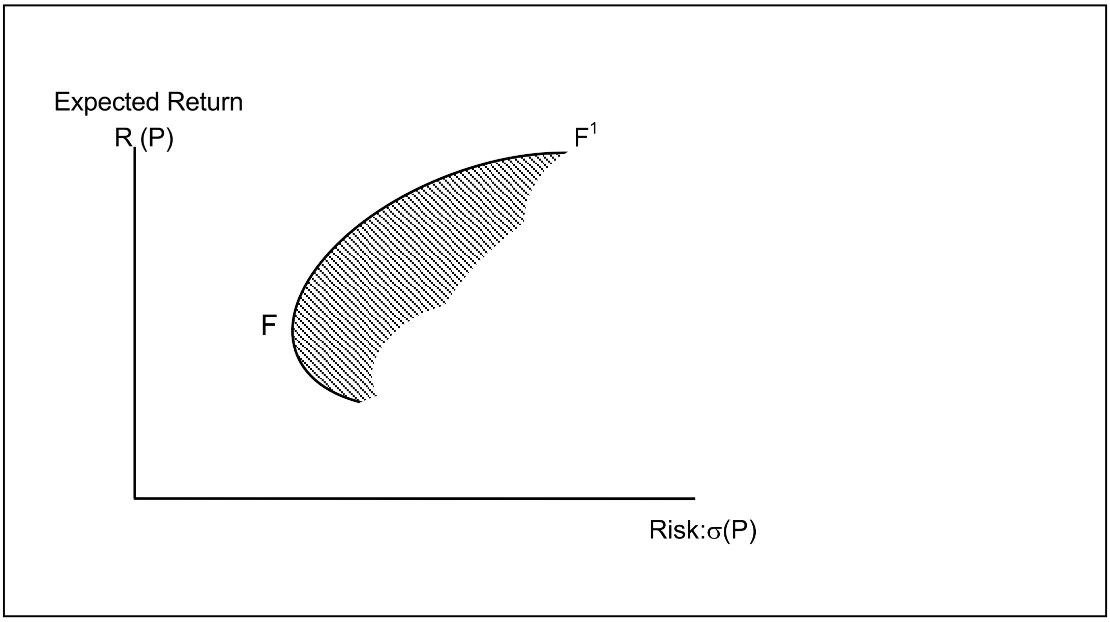

Figure 3.2: The Portfolio Efficiency Frontier: The Multi-Asset Case

The first point to note is that when an investment comprises a large number of assets instead of two, the possible portfolios now lie within on area, rather than along a line or curve. The area is constructed by plotting (infinitely) many lines or curves similar to those in Figure 3.1.

However, like a two-asset portfolio, rational, risk-averse investors or companies are not interested in all these possibilities, but only those that lie along the upper boundary between F and F^1. The portfolios that lie along this frontier are efficient because each produces the highest expected return for its given level of risk. To the right and below, alternative portfolios yield inferior results. To the left, no possibilities exist. Thus, an optimum portfolio for any investor can still be determined at an appropriate point on the efficiency frontier providing the individual's attitude toward risk is known.

So how is this calibrated?

3.5 The Optimum Portfolio

We have already observed that the *calculation* of statistical means and standard deviations is separate from their behavioural *interpretation*, which can create anomalies. For example, a particular problem we encountered within the context of investment appraisal was the "risk-return paradox" where one project offers a *lower return for less risk*, whilst the other offers a *higher return for greater risk*. Here, investor rationality (maximum return) and risk aversion (minimum variability) may be *insufficient* behavioural criteria for project selection. Similarly, with portfolio analysis:

> If two different portfolios lie on the efficiency frontier, it is impossible to choose between them without information on investor risk attitudes.

One solution is for the investor or company to consider a value for the portfolio's expected return R(P), say $R(p_i)$ depicted schematically in Figure 3.3.

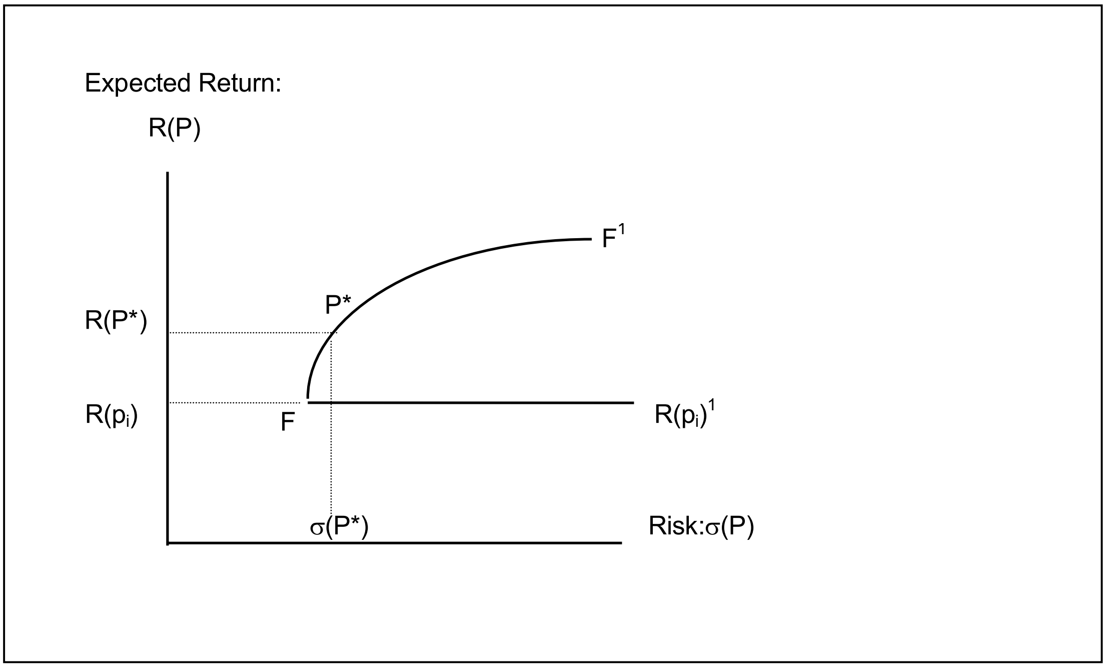

Figure 3.3: The Multi-Asset Efficiency Frontier and Investor Choice

All $R(p_i)$, $\sigma(p_i)$ combinations for different portfolio mixes are then represented by points along the horizontal line $R(p_i) - R(p_i)^1$ for which $R(P) = R(p_i)$. The leftmost point on this line, F then yields the portfolio investment mix that satisfies Equation (23) for our objective function:

$$\text{MIN: } \sigma(P), \text{ Given } R(P) = K \text{ (constant)}$$

By repeating the exercise for all other possible values of R(P) and obtaining every efficient value of $R(p_i)$ we can then trace the entire opportunity locus, $F\text{-}F^1$.The investor or company then subjectively select the investment combination yielding a maximum return, subject to the constraint imposed by the degree of risk they are willing to accept, say P* corresponding to R(P*) and s(P*) in the diagram.

Review Activity

As an optimisation procedure, the preceding model is theoretically sound. However, without today's computer technology and programming expertise, its practical application was a lengthy, repetitive process based on trial and error, when first developed in the 1950s. What investors and companies needed was a portfolio selection technique that actually incorporated their risk preferences into their analyses. Fortunately, there was a lifeline.

As we explained in the Summary and Conclusions of Chapter Two's Exercise text, (*PTFAE*) rational risk-averse investors, or companies, with a *two-asset* portfolio will always be willing to accept higher risk for a larger return, but *only up to a point*. Their precise cut-off rate is defined by an *indifference curve* that calibrates their risk attitude, based on the concept of *expected utility.*

We can apply this analysis to a *multi-asset* portfolio of investments. However, before we develop the mathematics, perhaps you might care to look back at Chapter Two (*PTFAE*)and the simple two-asset scenario before we continue.

In Chapter Two (*PTFAE*) we discovered that if an investor's or company's objective is to minimise the standard deviation of expected returns this can be determined by reference to a their utility *indifference curve,* which calibrates attitudes toward risk and return. Applied to portfolio analysis, the mathematical equation for any curve of indifference between portfolio risk and portfolio return for a rational investor can be written:

(24) $VAR(P) = \alpha + \lambda\ R(P)$

Graphically, the value of λ indicates the *steepness* of the curve and α indicates the *horizontal intercept*. Thus, the objective of the Markowitz portfolio model is to *minimise* α. If we rewrite Equation (24), for any indifference curve that relates to a portfolio containing n assets, this objective function is given by:

(25) MIN: $\alpha = VAR\ (P) - \lambda\ l\ R(P)$

For all possible values of $\lambda \geq 0$, where R(P) = K(constant), subject to the non-negativity constraints:

$\alpha_i \geq 0$, i = 1, 2, 3 …n

And the essential requirement that sources of funds equals uses and x_i be proportions expressed mathematically as:

$$\sum_{i=1}^{n} x_i = 1$$

Any portfolio that satisfies Equation (25) is *efficient* because no other asset combination will have a lower degree of risk for the requisite expected return.

An optimum portfolio for an individual investor is plotted in Figure 3.4. The efficiency frontier $F - F^1$ of risky portfolios still reveals that, to the right and below, alternative investments yield inferior results. To the left, no possibilities exist. However, we no longer determine an optimum portfolio for the investor by trial and error.

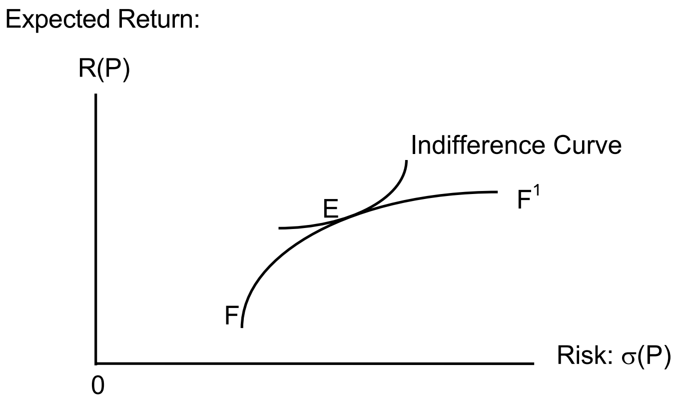

Figure 3.4: the Determination of an Optimum Portfolio: The Multi-Asset Case

The optimum portfolio *is at* the *point* where one of the curves for their equation of indifference (risk-return profile) is *tangential* to the frontier of efficient portfolios (point E on the curve $F\text{-}F^1$). This portfolio is optimal because it provides the best combination of risk and return to suit their preferences.

3.6 Summary and Conclusions

We have observed that the objective function of multi-asset portfolio analysis is represented by the following indifference equation.

$$(25)\ \text{MIN: } \alpha = \text{VAR(P)} - \lambda\ \text{R(P)}$$

This provides investors and companies with a standard, against which they can compare their preferred risk-return profile for any efficient portfolio.

However, its interpretation, like other portfolio equations throughout the Chapter assumes that the efficiency frontier has been correctly defined. Unfortunately, this in itself is no easy task.

Based upon the pioneering work of Markowitz (op. cit.) we explained how a rational and risk-averse investor, or company, in an efficient capital market (characterised by a normal distribution of returns) who require an optimal portfolio of investments can maximise utility, having regard to the relationship between the expected returns and their dispersion (risk) associated with the covariance of returns within a portfolio.

Any combination of investments produces a trade-off between the two statistical parameters; expected return and standard deviation (risk) associated with the covariability of individual returns. And according to Markowitz, this statistical analysis can be simplified.

> Efficient diversified portfolios are those which maximise return for a given level of risk, or minimise risk for a given level of return for different correlation coefficients.

The Markowitz portfolio selection model is theoretically sound. Unfortunately, even if we substitute the correlation coefficient into the covariance term of the portfolio variance, without the aid of computer software, the mathematical complexity of the variance-covariance matrix calculations associated with a multi-asset portfolio limits its applicability.

The *constraints* of Equation (25) are *linear* functions of the *n* variables x_i, whilst the *objective function* is an equation of the *second degree* in these variables. Consequently, methods of *quadratic* programming, rather than a simple *linear* programming calculation, must be employed by investors to *minimise* VAR(P) for various values of R (P) = K.

Once portfolio analysis extends beyond the two-asset case, the data requirements become increasingly formidable. If the covariance is used as a measure of the variability of returns, not only do we require estimates for the expected return and the variance for each asset in the portfolio but also estimates for the correlation matrix between the returns on all assets.

For example, if management invest equally in three projects, A, B and C, each deviation from the portfolio's expected return is given by:

$$[1/3\ r_iA - R(A)] + [1/3\ r_iB - R(B)] + [1/3\ r_iC - R(C)]$$

If the deviations are now squared to calculate the variance, the proportion 1/3 becomes $(1/3)^2$, so that:

$$\begin{aligned} VAR(P) &= VAR[1/3\ (A)+1/3\ (B)+1/3\ (C)] \\ &= (1/3)^2 \text{ (the sum of three variance terms, plus the sum of six covariances).} \end{aligned}$$

For a twenty asset portfolio:

$VAR(P) = (1/20)^2$ (sum of twenty variance, plus the sum of 380 covariances).

As a *general rule*, if there are $\Sigma\ x_i = n$ projects, we find that:

(26) VAR (P) = $(1/n)^2$ (sum of n variance terms, plus the sum of $n\ (n\text{-}1)$ covariances.

In the covariance matrix (x_i … x_n), x_i is paired in turn with each of the other projects (x_2 … x_n) making (n-1) pairs in total. Similarly, (n -1) pairs can be formed involving x_2 with each other x_i and so forth, through to x_n making n (n-1) permutations in total.

Of course, *half of these pairs will be duplicates*. The set x_1, x_2 is identical with x_2, x_1. The n asset case therefore requires only 1/2 $(n^2 - n)$ distinct covariance figures altogether, which represents a substantial data saving in relation to Equation (26). Nevertheless, the decision-maker's task is still daunting, as the number of investments for inclusion in a portfolio increases.

Not surprising, therefore, that without today's computer technology, a search began throughout the late 1950s and early 1960s for simpler mathematical and statistical measures of Markowitz portfolio risk and optimum asset selection, as the rest of our text will reveal.

3.7 Selected References

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4 The Market Portfolio

Introduction

The objective of efficient portfolio diversification is to achieve an overall standard deviation lower than that of its component parts without compromising overall return.

In an ideal world portfolio theory should enable:

- Investors (private or institutional) who play the stock market to model the effects of adding new securities to their existing spread.
- Companies to assess the extent to which the pattern of returns from new projects affects the risk of their existing operations.

For example, suppose there is a *perfect positive correlation* between two securities that comprise the market, or two products that comprise a firm's total investment. In other words, high and low returns always move sympathetically. It would pay the investor, or company, to place all their funds in whichever investment yields the highest return at the time. However, if there is *perfect inverse correlation*, where high returns on one investment are always associated with low returns on the other and *vice versa*, or there is *random (zero) correlation* between the returns, then it can be shown statistically that overall risk reduction can be achieved through diversification.

According to Markowitz (1952), if the correlation coefficient between any number of investments is less then one (perfect positive), the total risk of a portfolio measured by its standard deviation is lower than the weighted average of its constituent parts, with the greatest reduction reserved for a correlation coefficient of minus one (perfect inverse).

Thus, if the standard deviation of an individual investment is higher than that for a portfolio in which it is held, it would appear that some of the standard deviation must have been diversified away through correlation with other portfolio constituents, leaving a residual risk component associated with other factors.

Indeed, as we shall discover later, the reduction in *total* risk only relates to the *specific* risk associated with *micro-economic* factors, which are unique to individual sectors, companies, or projects. A proportion of *total* risk, termed *market* risk, based on *macro-economic* factors correlated with the market is inescapable.

The distinguishing features of specific and market risk had important consequences for the development of Markowitz efficiency and the emergence of Modern portfolio Theory (MPT) during the 1960's. For the moment, suffice it to say that whilst market risk is not diversifiable, theoretically, specific risk can be eliminated entirely if all rational investors diversify until they hold the *market portfolio*, which reflects the risk-return characteristics for every available financial security. In practice, this strategy is obviously unrealistic. But as we shall also discover later, studies have shown that with less than thirty diversified constituents it is feasible to reach a position where a portfolio's standard deviation is close to that for the market portfolio.

Of course, without today's computer technology and sophisticated software, there are still problems, as we observed in previous Chapters (*PTFA* and *PTFAE*). The significance of covariance terms in the Markowitz variance calculation are so unwieldy for a well-diversified risky portfolio that for most investors, with a global capital market to choose from, it is untenable. Even if we substitute the correlation coefficient into the covariance of the portfolio variance, the mathematical complexity of the variance-covariance matrix calculations for a risky multi-asset portfolio still limits its applicability. So, is there an alternative?

4.1 The Market Portfolio and Tobin's Theorem

We have already explained that if an individual or company objective is to minimize the standard deviation of an investment's expected return, this could be determined by reference to indifference curves, which calibrate attitudes toward risk and return. In Chapter Three (*PTFA*) and the summary of Chapter Two (*PTFAE*) we graphed an equation of *indifference* between portfolio risk and portfolio return for any rational investor relative to their optimum portfolio.

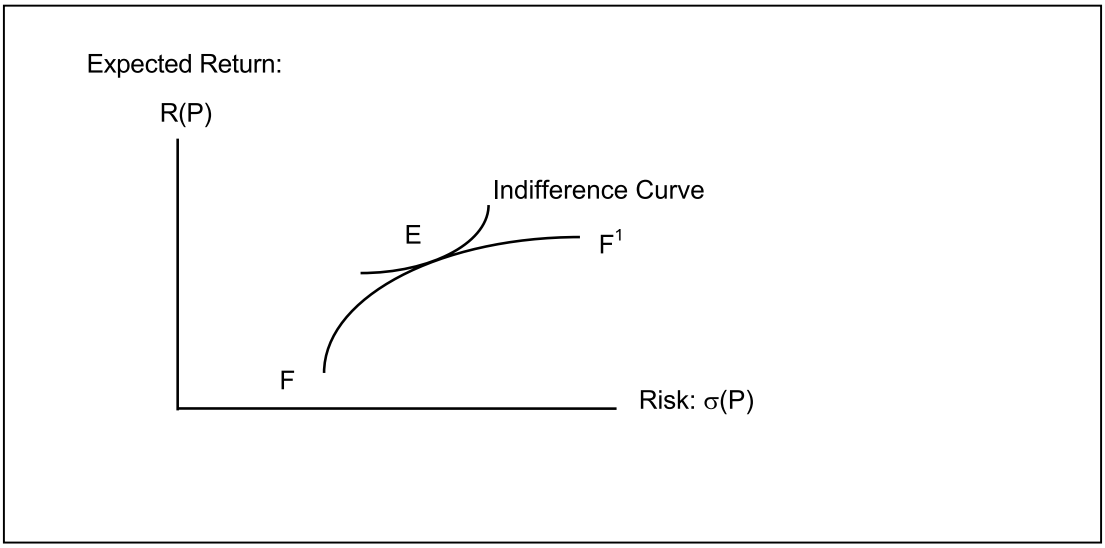

Figure 4.1: the Determination of an Optimum Portfolio: The Multi-Asset Case

Diagrammatically, you will recall that the *optimum* portfolio is determined at the *point* where one of the investor's indifference curves (risk-return profile) is *tangential* to the frontier of efficient portfolios. This portfolio (point E on the curve F-F[1] in Figure 4.1) is optimal because it provides the best combination of risk and return to suit their preferences.

However, apart from the computational difficulty of deriving optimum portfolios using variance-covariance matrix calculations (think 1950's theory without twenty-first century computer technology-software) this policy prescription only concerns *wholly* risky portfolios.

But what if *risk-free* investments (such as government stocks) are included in portfolios? Presumably, investors who are *totally* risk-averse would opt for a *riskless* selection of financial and government securities, including cash. Those who require an element of *liquidity* would construct a *mixed* portfolio that combines risk and risk-free investments to satisfy their needs.

Thus, what we require is a more sophisticated model than that initially offered by Markowitz, whereby the returns on new investments (risk-free or otherwise) can be compared with the risk of the market portfolio.

Fortunately, John Tobin (1958) developed such a model, built on Markowitz efficiency and the *perfect* capital market assumptions that underpin the Separation Theorem of Irving Fisher (1930) (with which you should be familiar).

Tobin demonstrates that in a perfect market where risky financial securities are traded with the option to lend or borrow at a risk-free rate, using risk-free assets, such as government securities.

> Investors and companies need not calculate a multiplicity of covariance terms. All they require is the covariance of a new investment's return with the overall return on the *efficient market portfolio.*

To understand what is now termed Tobin's *Separation Theorem*, suppose every stock market participant invests in all the market's risky securities, with their expenditure in each proportionate to the market's total capitalisation. Every investor's risky portfolio would now correspond to the market portfolio with a market return and market standard deviation, which we shall denote as M, r_m and s_m, respectively.

Tobin maintains that in *perfect* capital markets that are *efficient*, such an investment strategy is completely rational. In *equilibrium*, security prices will reflect their "true" intrinsic value. In other words, they provide a return commensurate with a degree of risk that justifies their inclusion in the market portfolio. Obviously, if a security's return does not compensate for risk; rational investors will want to sell their holding. But with no takers, price must fall and the yield will rise until the risk-return trade-off once again merits the security's inclusion in the market portfolio. Conversely, excess returns will lead to buying pressure that raises price and depresses yield as the security moves back into equilibrium.

This phenomenon is portrayed in Figure 4.2, where M represents the 100 per cent risky market portfolio, which lies along the efficiency frontier of all risky investment opportunities given by the curve $F\text{-}F^1$.

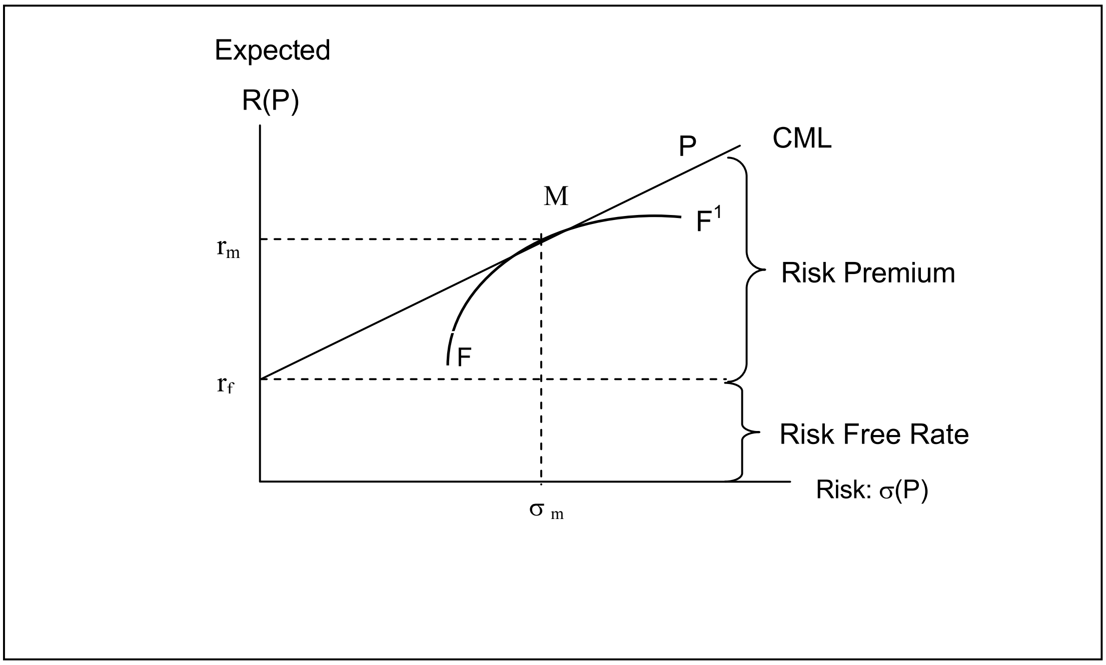

Figure 4.2: The Capital Market Line

Now, assume that all market participants can not only choose risky investments with the return r_m in the market portfolio M. They also have the option of investing in *risk-free* securities (such as short–term government stocks) at a risk-free rate, r_f. According to their aversion to risk and their desire for liquidity, we can now separate their preferences, (hence the term Separation Theorem). Investors may now opt for a *riskless* portfolio, or a *mixed* portfolio, which comprises any preferred combination of risk and risk-free securities.

Diagrammatically, investors can combine the market portfolio with risk-free investments to create a portfolio between r_f and M in Figure 4.2. If a line is drawn from the risk-free return r_f on the vertical axis of our diagram to the point of tangency with the efficiency frontier at point M, it is obvious that part of the original frontier (F-F^1) is now *inefficient*.

Below M, a higher return can be achieved for the same level of risk by combining the market portfolio with risk-free assets. Since rf denotes a riskless portfolio, the line rf -M represents increasing proportions of portfolio M combined with a reducing balance of investment at the risk-free rate.

Of course, as Fisher first explained way back in 1930, if capital markets are perfect (where borrowing and lending rates are equal) there is nothing to prevent individuals from borrowing at the risk-free rate to build up their investment portfolios. Tobin therefore adapted this concept to show that if investors could borrow at a risk-free rate and invest more in portfolio M using borrowed funds, they could construct a portfolio beyond M in Figure 4.2.

To show this, the line r_f to M has been extended to point P and beyond to CML. The effect eliminates the remainder of our original efficiency frontier. Any initial efficient portfolios lying along the curve $M\text{-}F_1$ are no longer desirable. With borrowing (leverage) there are always better portfolios with higher returns for the same risk. The line (r_f-M-CML) in Figure 4.2 is a new portfolio "efficiency frontier" for all investors, termed the *Capital Market Line* (CML).

Activity 1

To illustrate the purpose of the CML, let us assume that historically an investment company has *passively* held a market portfolio (M) of risky assets. This fund tracks the London FT-SE 100 (Footsie) on behalf of its clients.

However, with increasing global uncertainty the company now wishes to manage their portfolio *actively*, introducing risk-free investments into the mix and even borrowing funds if necessary.

Using Figure 4.2 for reference, briefly explain how the company's new strategy would redefine its optimum portfolio (or portfolios) if it is willing to borrow up to point P?

The portfolio lending-borrowing line (r_f-M-P) in Figure 4.2 is the new efficiency frontier (CML) for all the company's portfolio constituents. Portfolios lying along the CML between r_f and M are constructed by placing a proportion of their available funds in the market portfolio and the residual in risk-free assets. To establish a portfolio lying halfway up the line r_f-M, the company should divide funds equally between the two.

Portfolios lying along the CML beyond M (for example, P in the diagram) are constructed by placing all their funds in M, plus an amount borrowed at the risk-free rate (r_f). The amount borrowed would equal the ratio of the line r_f – M: M-P.

4.2 The CML and Quantitative Analyses

We have observed diagrammatically that if capital markets are efficient, all rational investors would ideally hold the market portfolio (M) irrespective of their risk attitudes. By finding the point of tangency between the efficiency frontier ($F\text{-}F^1$) and the capital market line (CML) then borrowing or lending at the risk-free rate (r_f) it is also possible for individual investors to achieve a desired balance between risk and return elsewhere on the CML.

Obviously, portfolios whose risk-return characteristics place it below the CML are inefficient and could be improved by altering their composition. It is also possible that an investor might "beat the market" (if only by luck rather than judgement) so that the portfolio's risk-return profile would lie above the CML, making it "super-efficient". However, if markets are efficient without access to insider information (as portfolio theorists assume) then this will be a temporary phenomenon.

Like the work of Fisher and Markowitz before him, Tobin's theorem is another landmark in the development of financial theory, which you ought to read at source. At the very least you need to be able to manipulate the following statistical equations, which we shall apply to an Exercise in Chapter Four of our companion text (*PTFAE*).

Portfolio Risk

So, let us begin by *redefining* our general portfolio risk formula based on Equation (7) for the standard deviation (which you first encountered in Chapter Two). Combining the market variance of returns (σ^2_m) with the variance of risk-free investments (σ^2_f):

(27) $\sigma_p = \sqrt{} \ [x^2\sigma^2_m + (1-x)^2 \ \sigma^2_f + 2x \ (1-x) \ \sigma_m \ \sigma_f \ \mathrm{COR} \ (_{m,f})]$

The first point to note is that because the variability of risk-free returns is obviously zero, their variance (σ^2_f) and standard deviation (σ_f) equals zero. The second and third terms of Equation (27), which define the variance of the risk-free investment and the correlation coefficient, disappear completely. Thus, Equation (27) for the portfolio's standard deviation simplifies to:

(28) $\sigma_p = \sqrt{} \ (x^2 \ \sigma^2_m)$

Rearranging the terms of Equation (28) with only one unknown and simplifying, we can also determine the proportion of funds (x) invested in the market portfolio. Given any investor's preferred portfolio and the market standard deviation of returns (σ_p and σ_m):

(28) $x = \sigma_p / \sigma_m$

Portfolio Return

In Chapter Two we defined the expected return for a two-asset portfolio R(P) as the weighted average of expected returns from two investments or projects, R(A) and R(B), where the weights are the proportional funds invested in each. Mathematically, this is given by:

(1) $R(P) = x\, R(A) + (1 - x)\, R(B)$

The equation can be adapted to calculate the expected return (r_p) for any portfolio that includes a combination of risky and risk-free investments, whose returns are (r_m) and (r_f) respectively.

(29) $r_p = x\, r_m + (1-x)\, r_f$

The Market Price of Risk or Risk Premium

Because the CML is a *simple linear regression* line, its slope (α_m) is a *constant*, measured by:

(30) $\alpha_m = (r_m - r_f) / \sigma_m$

The expected return for any portfolio on the CML (r_p) can also be expressed as:

(31) $r_p = r_f + [(r_m - r_f) / \sigma_m]\, \sigma_p$

Given r_f (the risk-free rate of return) which is the *intercept* illustrated in Figure 4.2 (where σ_p equals zero) r_m is still the market portfolio return and σ_m and σ_p define market risk and the risk of the particular portfolio, respectively.

The *constant* slope of the CML (α_m) defined by Equation (30) is called the *market price of risk*. It represents the *incremental* return ($r_m - r_f$) obtained by investing in the market portfolio (M) divided by market risk (σ_m). In effect it is the *risk premium* added to the risk-free rate (sketched in Figure 4.2) to establish the total return for any particular portfolio's risk-return trade off.

For example, with a risk premium α_m defined by Equation (30), the incremental return from a portfolio bearing risk (σ_p) in relation to market risk (σ_m) is given by:

$$\alpha_m (\sigma_p - \sigma_m)$$

This can be confirmed if we were to compare a particular portfolio return with that for the market portfolio. The difference between the two ($r_p - r_m$) equals the market price of risk (α_m) times the spread ($\sigma_p - \sigma_m$).

To summarise, the expected return for any efficient portfolio lying on the CML comprising the market portfolio, plus either borrowing or lending at the risk free rate can be expressed by simplifying Equations (30) and (31), so that:

$$(32)\ r_p = r_f + \alpha m\ \sigma_p$$

In other words, the expected return of an efficient portfolio (rp) equals the risk-free rate of return (r_f) plus a risk premium ($\alpha_m.\sigma_p$).This premium reflects the market's risk-return trade-off (α_m) combined with the portfolio's own risk (σ_p).

4.3 Systematic and Unsystematic Risk

The objective of portfolio diversification is the selection of investment opportunities that reduce *total* portfolio risk without compromising *overall* return. The preceding analysis based on Markowitz efficiency and Tobin's Separation Theorem in perfect capital markets indicates that:

If the standard deviation (risk) of an individual investment is higher than that of the portfolio in which it is held, then part of the standard deviation must have been diversified away through correlation with other portfolio constituents.

A high level of diversification results in rational investors holding the market portfolio, which they will do in combination with lending or borrowing at the risk-free rate. This leaves only the element of risk that is correlated with the market as a whole. In other words portfolio risk equals market risk, which is undiversifiable

$$\sigma_p - \sigma_m$$

To clarify this point for future analysis, Figure 4.3 summarises the relationship between total risk and its component parts where.

Total risk is split between:

- *Systematic* or *market risk*, so called because it is endemic throughout the system (market) and is undiversifiable. It relates to general economic factors that affect all firms and financial securities, and explains why share prices tend to move in sympathy.
- *Unsystematic risk,* sometimes termed *specific*, *residual*, or *unique risk*, relates to specific (unique) economic factors, which impact upon individual industries, companies, securities and projects. It can be eliminated entirely through efficient diversification.

In terms of our earlier analysis, systematic risk measures the extent to which an investment's return moves sympathetically (systematically) with all the financial securities that comprise the market portfolio (the *system*). It describes a particular portfolio's inherent sensitivity to global political and macro-economic volatility. The best recent example, of course, is the 2007 financial meltdown and subsequent economic recession. Because individual companies or investors have no control over such events, they require a rate of return commensurate with their relative systematic risk. The greater this risk, the higher the rate of return required by those with widely diversified portfolios that reflect movements in the market as a whole.

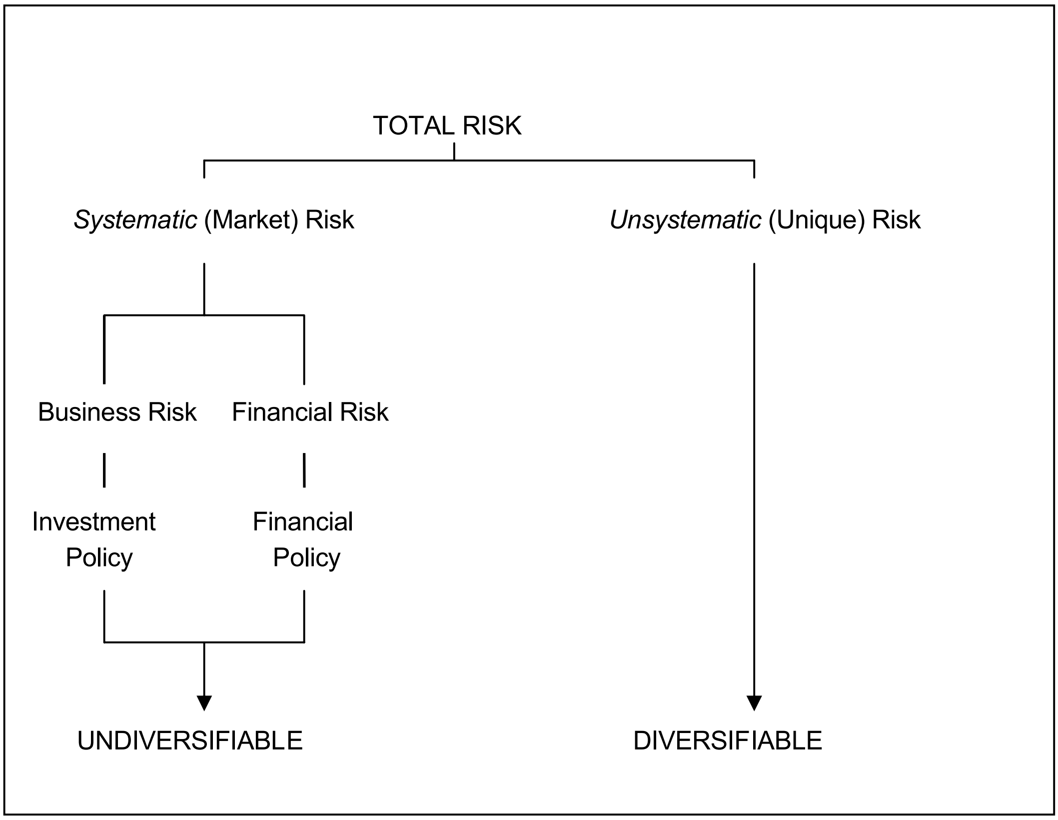

Figure 4.3: The Inter-relationship of Risk Concepts

In contrast, unsystematic risk relates to an individual security's price or even a project and is independent of market risk. Applied to individual companies, it is caused by micro-economic factors such as the level of profitability, product innovation and the quality of financial management. Because it is completely diversifiable (variations in returns cancel out over time) unsystematic risk carries no market premium. Thus, all the risk in a fully diversified portfolio is market or systematic risk.

You may have encountered systematic risk elsewhere in your studies under other names. For example, Figure 4.3 reveals that systematic risk comprises a company's *business risk* and possibly *financial risk.* Certainly if you have read the author's other *SFM* texts, you will recall that business risk reflects the unavoidable variability of project returns according to the nature of the investment (*investment policy*). This may be higher or lower than that for other projects, or the market as a whole. Systematic risk may also reflect a premium for financial risk, which arises from the proportion of debt to equity in a firm's capital structure (gearing) and the amount of dividends paid in relation to the level of retained earnings, (*financial policy*). Of course, there is considerable empirical support for the view that financial risk is irrelevant based on the seminal work of Modigliani-Miller (1958 and 1961) explained in *SFM*. Irrespective of whether financial policies matter, for the moment all we need say is that for all-equity firms with full dividend distribution policies, there is an academic consensus that business risk equals systematic (market) risk and is not diversifiable.

Review Activity

Given our analysis of Markowitz efficiency and the Separation of Tobin, briefly summarise the implications for optimum portfolio management?

4.4 Summary and Conclusions

Markowitz, explains how investors or companies can reduce risk but maintain their return by holding more than one investment providing their returns are not positively correlated. This implies that all rational investors will diversify their risky investments into a portfolio.

Tobin illustrates how the introduction of risk-free investments further reduces portfolio risk, using the CML to define a new frontier of efficient portfolios.

Consequently, all investors are capable of eliminating unsystematic risk by expanding their investment portfolios until they reflect the market portfolio.

Based on numerous studies, Figure 4.4 highlights the empirical fact that up to 95 percent of unsystematic risk can be diversified away by randomly increasing the number of investments in a portfolio to about thirty. With one investment, portfolio risk is represented by the sum of unsystematic and systematic risk, i.e. the investment's *total risk* as measured by its standard deviation. When the portfolio constituents reach double figures virtually all the risk associated with holding that portfolio becomes systematic or market risk. See Fisher and Lorie (1970) for one of the earliest and best reviews of the phenomenon.

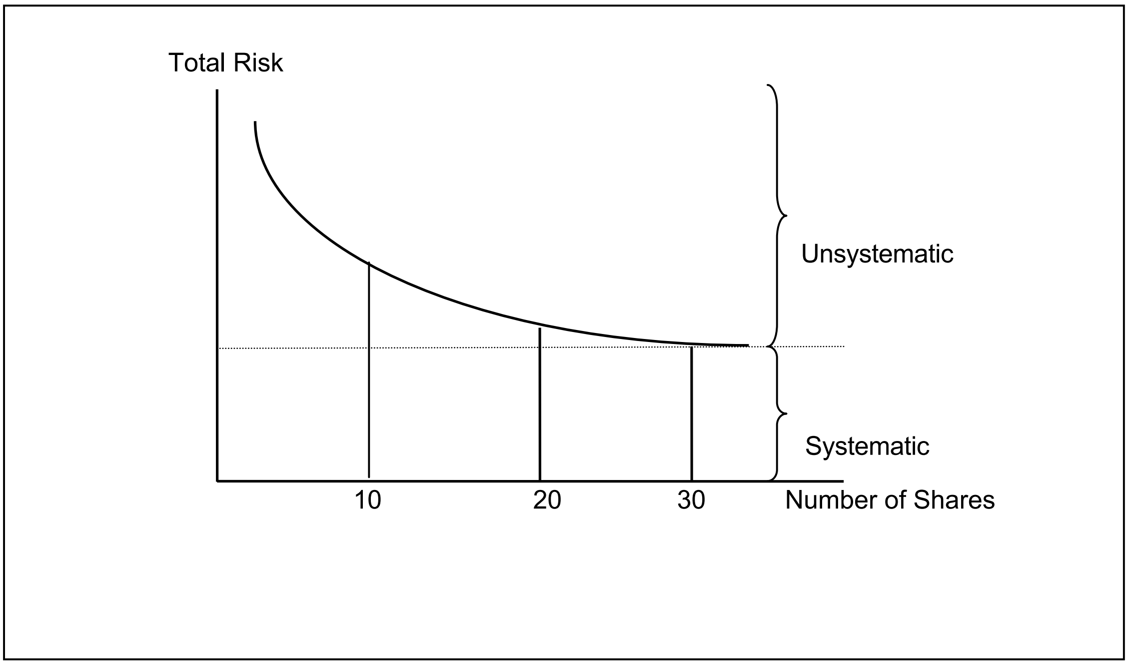

Figure 4.4: Portfolio Risk and Diversification

It should therefore come as no surprise that without to-days computer technology and software to solve their problems:

> Academic and financial analysts of the 1960's, requiring a much simpler model than that offered by Markowitz to enable them to diversify efficiently, were quick to appreciate the work of Tobin and the utility of the relationship between the systematic risk of either a project, a financial security, or a portfolio and their returns.

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Part III:
Models Of Capital Asset Pricing

5 The Beta Factor

Introduction

In an ideal world, the portfolio theory of Markowitz (1952) should provide management with a practical model for measuring the extent to which the pattern of returns from a new project affects the risk of a firm's existing operations. For those playing the stock market, portfolio analysis should also reveal the effects of adding new securities to an existing spread. The objective of efficient portfolio diversification is to achieve an overall standard deviation lower than that of its component parts without compromising overall return.

Unfortunately, as we observed in Part Two, the calculation of the covariance terms in the risk (variance) equation becomes unwieldy as the number of portfolio constituents increase. So much so, that without today's computer technology and software, the operational utility of the basic model is severely limited. Academic contemporaries of Markowitz therefore sought alternative ways to measure investment risk

This began with the realisation that the *total risk* of an investment (the standard deviation of its returns) within a diversified portfolio can be divided into *systematic* and *unsystematic* risk. You will recall that the latter can be eliminated entirely by efficient diversification. The other (also termed *market* risk) cannot. It therefore affects the overall risk of the portfolio in which the investment is included.

Since all rational investors (including management) interested in wealth maximisation should be concerned with individual security (or project) risk relative to the stock market as a whole, portfolio analysts were quick to appreciate the importance of systematic (market) risk. According to Tobin (1958) it represents the only risk that they will pay a premium to avoid.

Using this information and the assumptions of perfect markets with opportunities for risk-free investment, the required return on a risky investment was therefore redefined as the risk-free return, plus a premium for risk. This premium is not determined by the total risk of the investment, but only by its systematic (market) risk.

Of course, the systematic risk of an individual financial security (a company's share, say) might be higher or lower than the overall risk of the market within which it is listed. Likewise, the systematic risk for some projects may differ from others within an individual company. And this is where the theoretical development of the beta factor (b) and the Capital Asset Pricing Model (CAPM) fit into portfolio analysis.

We shall begin Part Three by defining the relationship between an individual investment's systematic risk and market risk measured by (b_j) its *beta* factor (or coefficient). Using our *earlier notation* and continuing with the *equation numbering* from previous Chapters, which ended with Equation (32):

(33) $$\beta_j = \frac{COV(j,m)}{VAR(m)}$$

This factor equals the covariance of an investment's return, relative to the market portfolio, divided by the variance of that portfolio.

As we shall discover, beta factors exhibit the following characteristics:

The market as a whole has a b = 1
A risk-free security has a b = 0
A security with systematic risk below the market average has a b < 1
A security with systematic risk above the market average has a b > 1
A security with systematic risk equal to the market average has a b = 1

The significance of a security's β value for the purpose of stock market investment is quite straightforward. If overall returns are expected to fall (*a bear* market) it is worth buying securities with low β values because they are expected to fall less than the market. Conversely, if returns are expected to rise generally (a *bull* scenario) it is worth buying securities with high β values because they should rise faster than the market.

Ideally, beta factors should reflect *expectations* about the *future* responsiveness of security (or project) returns to corresponding changes in the market. However, without this information, we shall explain how individual returns can be compared with the market by plotting a *linear* regression line through *historical* data.

Armed with an operational measure for the market price of risk (β), in Chapter Six we shall explain the rationale for the Capital Asset Pricing Model (CAPM) as an alternative to Markowitz theory for constructing efficient portfolios.

For any investment with a beta of β_j, its expected return is given by the CAPM *equation*:

(34) $r_j = r_f + (r_m - r_f)\ \beta_j$

Similarly, because all the characteristics of systematic betas apply to a *portfolio*, as well as an *individual* security, any portfolio return (r_p) with a portfolio beta (β_p) can be defined as:

(35) $r_p = r_f + (r_m - r_f)\ \beta_p$

For a given a level of systematic risk, the CAPM determines the expected rate of return for any investment relative to its beta value. This equals the risk-free rate of interest, *plus* the product of a market risk premium and the investment's beta coefficient. For example, the mean return on equity that provides adequate compensation for holding a share is the value obtained by incorporating the appropriate equity beta into the CAPM equation.

The CAPM can be used to estimate the expected return on a security, portfolio, or project, by investors, or management, who desire to eliminate unsystematic risk through efficient diversification and assess the required return for a given level of non-diversifiable, systematic (market) risk. As a consequence, they can tailor their portfolio of investments to suit their individual risk-return (utility) profiles.

Finally, in Chapter Six we shall validate the CAPM by reviewing the balance of empirical evidence for its application within the context of capital markets.

In Chapter Seven we shall then focus on the CAPM's operational relevance for strategic financial management within a corporate capital budgeting framework, characterised by capital gearing. And as we shall explain, the stock market CAPM can be modified to derive a project discount rate based on the systematic risk of an individual investment. Moreover, it can be used to compare different projects across different risk classes.

At the end of Part Three, by cross-referencing this text and its companion Exercises (underpinned by *SFM* and *SFME* material from *Bookboon*) you should therefore be able to confirm that:

> The CAPM not only represents a viable alternative to managerial investment appraisal techniques using NPV wealth maximisation, mean-variance analysis, expected utility models and the WACC concept. It also establishes a mathematical connection with the seminal leverage theories of Modigliani and Miller (MM 1958 and 1961).

5.1 Beta, Systemic Risk and the Characteristic Line

Suppose the price of a share selected for inclusion in a portfolio happens to increase when the equity market rises. Of prime concern to investors is the extent to which the share's total price increased because of unsystematic (specific) risk, which is diversifiable, rather than systematic (market) risk that is not.

A practical solution to the problem is to isolate systemic risk by comparing past trends between individual share price movements with movements in the market as a whole, using an appropriate all-share stock market index.

So, we could plot a "scatter" diagram that correlates percentage movements for:

- The selected share price, on the vertical axis,
- Overall market prices using a relevant index on the horizontal axis.

The "spread" of observations equals unsystematic risk. Our line of "best fit" represents systematic risk determined by *regressing* historical share prices against the overall market over the time period. Using the statistical method of *least squares*, this linear regression is termed the share's *Characteristic Line.*

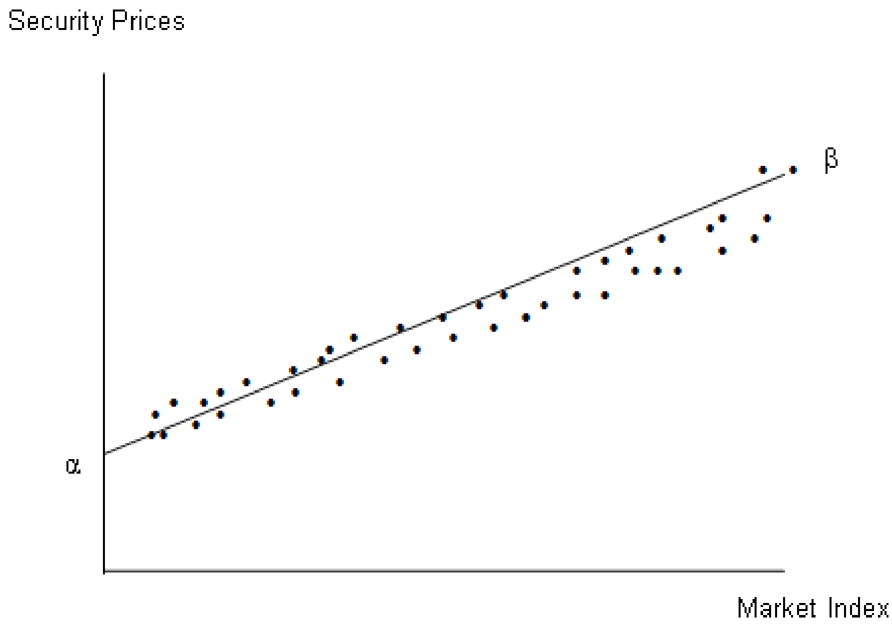

Figure 5.1: The Relationship between Security Prices and Market Movements The Characteristic Line

As Figure 5.1 reveals, the *vertical intercept* of the regression line, termed the *alpha factor* (α) measures the average percentage movement in share price if there is no movement in the market. It represents the amount by which an individual share price is greater or less than the market's systemic risk would lead us to expect. A positive alpha indicates that a share has outperformed the market and *vice versa*.

The *slope* of our regression line in relation to the horizontal axis is the *beta factor* (b) measured by the share's covariance with the market (rather than individual securities) divided by the variance of the market. This calibrates the *volatility* of an individual share price relative to market movements, (more of which later). For the moment, suffice it to say that the steeper the Characteristic Line the more volatile the share's performance and the higher its systematic risk. Moreover, if the slope of the Characteristic Line is very steep, b will be greater than 1.0. The security's performance is volatile and the systematic risk is high. If we performed a similar analysis for another security, the line might be very shallow. In this case, the security will have a low degree of systematic risk. It is far less volatile than the market portfolio and b will be less than 1.0. Needless to say, when b equals 1.0 then a security's price has "tracked" the market as a whole and exhibits *zero* volatility.

The beta factor has two further convenient statistical properties applicable to investors generally and management in particular.

First, it is a far simpler, computational proxy for the covariance (relative risk) in our original Markowitz portfolio model. Instead of generating numerous new covariance terms, when portfolio constituents (securities-projects) increase with diversification, all we require is the covariance on the additional investment relative to the efficient market portfolio.

Second, the Characteristic Line applies to investment *returns*, as well as *prices*. All risky investments with a market price must have an expected return associated with risk, which justify their inclusion within the market portfolio that all risky investors are willing to hold.

Activity 1

If you read different financial texts, the presentation of the Characteristic Line is a common source of confusion. Authors often define the axes differently, sometimes with prices and sometimes returns.

Consider Figure 5.2, where *returns* have been substituted for the *prices* of Figure 5.1. Does this affect our linear interpretation of alpha and beta?

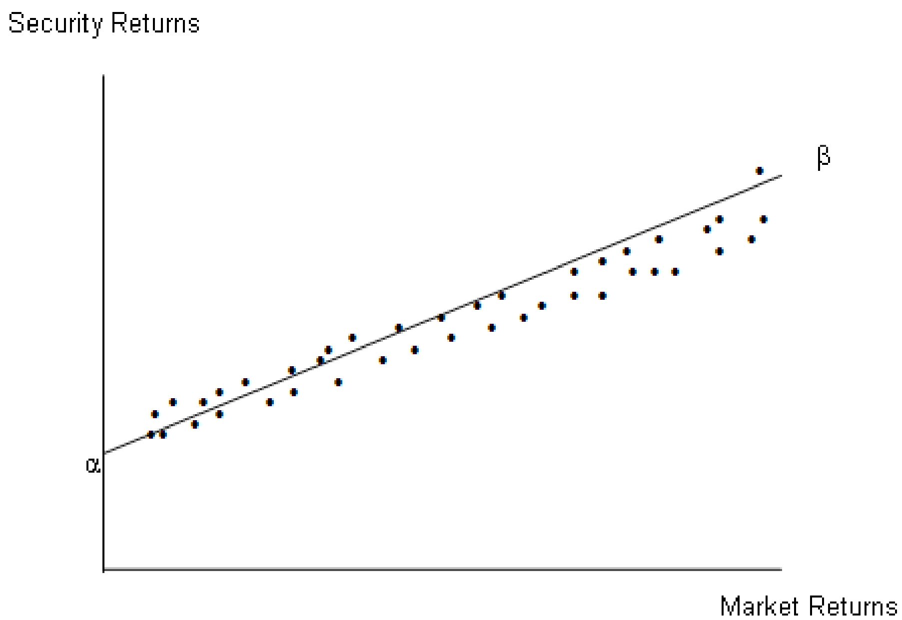

Figure 5.2: The Relationship between Security Returns and Market Returns The Characteristic Line

The substitution of returns for prices in the regression doesn't affect our interpretation of the graph, because returns obviously determine prices.

- The horizontal intercept (α) now measures the extent to which *returns* on an investment are greater or less than those for the market portfolio.
- The steeper the slope of the Characteristic Line, then the more volatile the return, the higher the systematic risk (b) and *vice versa*.

We began by graphing the security prices of risky investments and total market capitalisation using a stock market index because it serves to remind us that the development of Capital Market Theory initially arose from portfolio theory as a pricing model. However, because theorists discovered that returns (like prices) can also be correlated to the market, with important consequences for internal management decision making, as well as stock market investment, many modern texts focus on returns and skip pricing theory altogether.

Henceforth, we too, shall place increasing emphasis on returns to set the scene for Chapter Seven. There our ultimate concern will relate to strategic financial management and an optimum project selection process derived from models of capital asset pricing using b factors for individual companies that provide the highest expected return in terms of investor attitudes to the risk involved.

5.2 The Mathematical Derivation of Beta

So far, we have only explained a beta factor (b) by reference to a *graphical* relationship between the pricing or return of an individual security's risk and overall market risk. Let us now derive *mathematical* formulae for b by adapting our *earlier notation* and continuing with the *equation numbering* from previous Chapters. This ended with Equation (32) and began with Equation (33) in our Introduction the present one.

Suppose an individual was to place all their investment funds in all the financial securities that comprise the global stock market in proportion to the individual value of each constituent relative to the market's total value.

The market portfolio has a variance of VAR(m) and the covariance of an individual security j with the market average is COV(j,m). So, the relative risk (the security's beta) denoted by β_j is given by our earlier equation:

$$(33)\ \beta_j = \frac{COV(j,m)}{VAR(m)}$$

Alternatively, we know from Chapter Two that given the relationship between the covariance and the *linear* correlation coefficient, the covariance term in Equation (33) can be rewritten as:

$$COV\ (j,m) = COR\ (j,m)\ .\ \sigma\, j\ \sigma\, m$$

So, we can also define a theoretical value for beta as follows:

$$\beta_j = \frac{COR(j,m)\ .\ \sigma\, j\ \sigma\, m}{\sigma^2(m)}$$

And simplifying, (allowing for the equation numbering in our Introduction to this Chapter):

$$(36)\ \beta_j = \frac{COR(j\ m)\ \sigma\, j}{\sigma\ (m)}$$

If information on the variance or standard deviation and covariance or correlation coefficient is readily available, the calculation of beta is extremely straightforward using either equation. Ideally, we should determine b using *forecast* data (in order to appraise *future* investments). In its absence, however, we can derive an *estimator* using least-squares regression. This plots a security's *historical* periodic return against the corresponding return for the appropriate market index. For example, an ordinary share's return r_t (common stock) is given by:

$$r_t = \frac{\text{Increase in the period's ex-div value per share + the dividend per share paid}}{\text{Share value at the beginning of the period}}$$

Obviously it needs to be adjusted for events such as bonus or rights issues and any capital reorganisation-reconstruction. Fortunately, because of their ease of calculation, β estimators are published regularly by the financial services industry for stock exchange listings world-wide. A particularly fine example is the London Business School Risk Management Service (LBSRMS) that supplies details of equity betas, which are also geared up (leveraged) according to the firm's capital structure (more of which later in Chapter Seven).

Given the universal, freely available publication of beta factors, considerable empirical research on their behaviour has been undertaken over a long period of time. So much so, that as a measure of systematic risk they are now known to exhibit another extremely convenient property (which also explains their popularity within the investment community).

Although alpha risk varies considerably over time, numerous studies (beginning with Black, Jensen and Scholes in 1972) have continually shown that beta values are more stable. They move only slowly and display a near *straight-line* relationship with their returns. The longer the period analysed, the better. The more data analysed, the better. Thus, betas are invaluable for efficient portfolio selection. Investors can tailor a portfolio to their specific risk-return (utility) requirements, aiming to hold *aggressive* stocks with a b in excess of one while the market is rising, and less than one (*defensive*) when the market is falling.

Activity 2

Explain the investment implications of a beta factor of 1.15 and a beta factor that is less than the market portfolio

A beta of 1.15 implies that if the underlying market with a beta factor of one were to rise by 10 per cent, then the stock may be expected to rise by 11.5 per cent. Conversely, a security with a beta of less than one would not be as responsive to market movements. In this situation, smaller systemic risk would mean that investors would be satisfied with a return that is below the market average. The market portfolio has a beta of one precisely because the covariance of the market portfolio with itself is identical to the variance of the market portfolio. Needless to say, a risk-free investment has a beta of zero because its covariance with the market is zero.

5.3 The Security Market Line

Let us pause for thought:

- *Total* risk comprises unsystematic and systematic risk.
- *Unsystematic* risk, unique to each company, can be eliminated by portfolio diversification.
- *Systematic* risk is undiversifiable and depends on the market as a whole.

These distinctions between total, unsystematic and systematic risk are vital to our understanding of the development of Modern Portfolio Theory (MPT). Not only do they validate beta factors as a measure of the only risk that investors will pay a premium to avoid. As we shall discover, they also explain the rationale for the Capital Asset Pricing Model (CAPM) whereby investors can assess the portfolio returns that satisfy their risk-return requirements. So, before we consider the CAPM in detail, let us contrast systemic beta analysis with basic portfolio theory that only considers total risk.

The linear relationship between *total* portfolio risk and expected returns, the *Capital Market Line* (CML) based on Markowitz efficiency and Tobin's Theorem, graphed in Chapter Four does not hold for *individual* risky investments. Conversely, all the characteristics of systemic beta risk apply to portfolios *and* individual securities. The beta of a portfolio is simply the weighted average of the beta factors of its constituents.

This new relationship becomes clear if we reconstruct the CML (Figures 4.2 and 4.1 from Chapter Four of our Theory and Exercise texts, respectively) to form what is termed the *Security Market Line* (SML). As Figure 5.3 illustrates, the expected return is still calibrated on the vertical axis but the SML substitutes systemic risk (β) for total risk (σ_p) on the horizontal axis of our earlier CML diagrams.

Once beta factors are calculated (not a problem) the SML provides a universal measure of risk that still adheres to *Markowitz efficiency* and his criteria for portfolio selection, namely:

Maximise return for a given level of risk Minimise risk for a given level of return

Like the CML, the SML still confirms that the *optimum* portfolio is the *market* portfolio. Because the return on a portfolio (or security) depends on whether it follows market prices as a whole, the closer the correlation between a portfolio (security) and the market index, then the greater will be its expected return. Finally, the SML predicts that both portfolios and securities with higher beta values will have higher returns and *vice versa*.

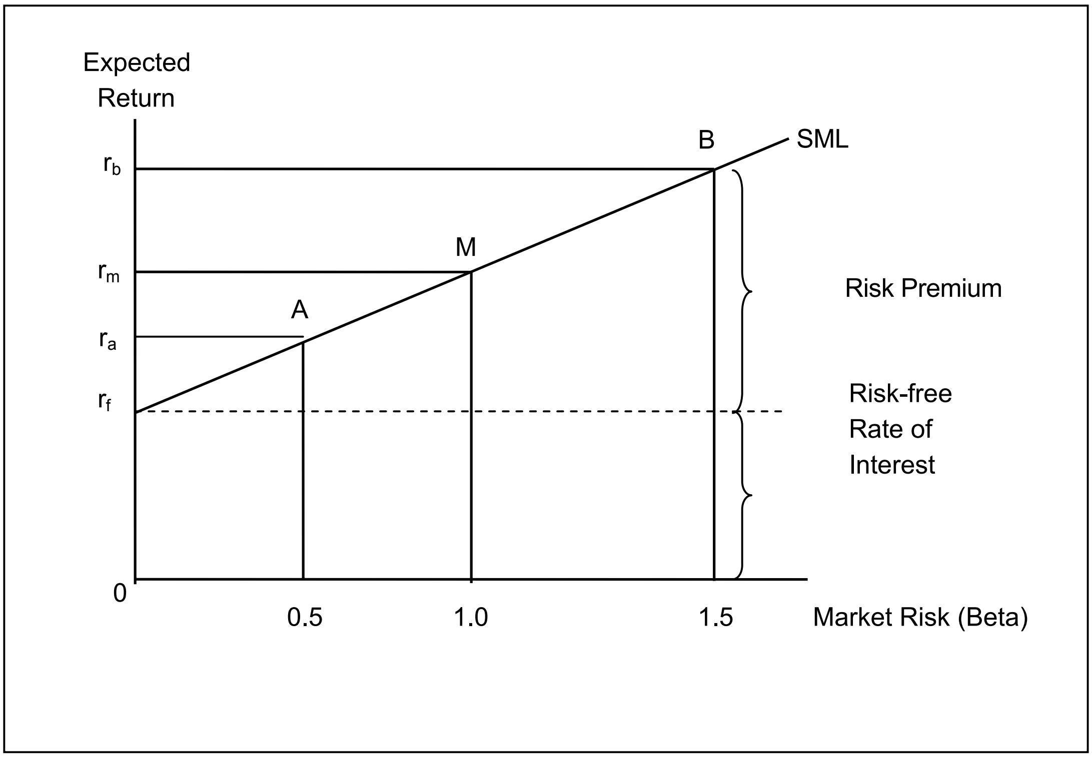

Figure 5.3: The Security Market Line

As Figure 5.3 illustrates, the expected risk-rate return of r_m from a balanced market portfolio (M) will correspond to a beta value of one, since the portfolio cannot be more or less risky than the market as a whole. The expected return on risk-free investment (r_f) obviously exhibits a beta value of zero.

Portfolio A (or anywhere on the line r_f-M) represents a *lending* portfolio with a mixture of risk and risk-free securities. Portfolio B is a *borrowing* or leveraged portfolio, because beyond (M) additional securities are purchased by borrowing at the risk-free rate of interest.

Review Activity

Given your knowledge of perfect capital markets, Fisher's Separation Theorem, stock market efficiency, mean-variance analysis, utility theory, Markowitz efficiency and Tobin's Capital Market Line (CML):

Briefly summarise what the Security Market Line (SML) offers rational, risk-averse individuals seeking a well-diversified portfolio of investments?

5.4 Summary and Conclusions

Throughout our analyses (including the background *SFM* and *SFME* texts) we have observed how rational, risk-averse individuals and companies operating in perfect markets with no "barriers to trade" can rank *individual* investments by interpreting their expected returns and standard deviations using the concept of expected utility to calibrate their risk-return attitudes. In this book (and its Exercise companion) we began with the same mean-variance efficiency criteria to derive optimum *portfolio* investments that can reduce risk (standard deviation) without impairing return. In Part Two this culminated with Tobin's Theorem and the CML that incorporates borrowing and lending opportunities to define optimum "efficient" portfolio investment opportunities.

Unfortunately, the CML only calibrates total risk (σ_p) not all of which is diversifiable. Fortunately, the SML offers investors a lifeline, by discriminating between non-systemic and systemic risk. The latter is defined by a beta factor that measures relative (systematic) risk, which explains how rational investors with different utility (risk-return) requirements can choose an optimum portfolio by borrowing or lending at the risk-free rate.

We shall return to this topic in Chapter Six when risk is related to the expected return from an investment or portfolio using the CAPM.

5.5 Selected References

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6 The Capital Asset Pricing Model (Capm)

Introduction

Basic portfolio theory defines the expected return from a risky investment in general terms as the risk-free return, plus a premium for risk. However, we have observed that this premium is determined not by the overall risk of the investment but only by its systematic (market) risk.

(36) $\beta_j = \frac{COR(j\ m)\ \sigma\ j}{\sigma\ (m)}$

Using the geometry of the Security Market Line (SML) that determines the market risk premium (b), numerous academics, notably Sharpe (1963) followed by Lintner (1965), Treynor (1965) and Mossin (1966) were quick to develop (quite independently) the *Capital Asset Pricing Model* (CAPM) as a logical extension to basic portfolio theory.

Today, the CAPM is regarded by many as a superior model of security price behaviour to others based on wealth maximisation criteria with which you should be familiar. For example, unlike the dividend and earnings share valuation models of Gordon (1962) and Modigliani and Miller (1961) covered in our *SFM* and *SFME* texts, the CAPM explicitly identifies the risk associated with an ordinary share (common stock) as well as the future returns it is expected to generate. Moreover, the CAPM can also express investment returns in two forms

For individual securities:

(34) $r_j = r_f + (r_m - r_f)\ \beta_j$

And because systemic betas apply to a *portfolio*, as well as an *individual* investment:

(35) $r_p = r_f + (r_m - r_f)\ \beta_p$

For a given a level of systematic risk, the CAPM determines the expected rate of return for any investment (security, project, or portfolio) relative to its beta value defined by the SML (a market index). As we shall discover, it also establishes whether individual securities, projects (or their portfolios) are under or over-priced relative to the market, (hence its name).The CAPM can therefore be used by investors or management, who desire to eliminate unsystematic risk through efficient diversification and assess the required return for a given level of non-diversifiable, systematic (market) risk. As a consequence, they can tailor their portfolio of investments to suit their individual risk- return (utility) profiles.

6.1 The CAPM Assumptions

The CAPM is a *single-period* model, which means that all investors make the same decision over the same time horizon. Expected returns arise from expectations over the same period.

The CAPM is a *single-index* model because systemic risk is prescribed entirely by *one* factor; the beta factor.

The CAPM is defined by random variables that are normally distributed, characterised by mean expected returns and covariances, upon which all investors agree.

Markowitz mean-variance efficiency criteria based on perfect markets still determine the optimum portfolio (P).

$$\text{MAX: } R(P), \text{ given } \delta(P)$$
$$\text{MIN: } \delta(P), \text{ given } R(P).$$

- All investments are infinitely divisible.
- All investors are rational and risk averse.
- All investors are price takers, since no individual, firm or financial institution is large enough to distort prevailing market values.
- All investors can borrow-lend without restriction at the risk-free market rate of interest.
- Transaction costs are zero and the tax system is neutral.
- There is a perfect capital market where all information is available and costless.

Table 6.1: The CAPM Assumptions

The application of the CAPM and beta factors is straight forward as far as stock market tactics are concerned. The model assumes that investors have three options when managing a portfolio:

1) To trade,
2) To hold,
3) To substitute, (*i.e*. securities for property, property for cash, cash for gold *etc.*).

A profitable trade is accomplished by buying (selling), undervalued (overvalued) securities relative to an appropriate measure of systematic risk, a global stock market index such as the FT/ S&P World Index. If the market is "bullish" and prices are expected to rise generally, it is worth buying securities with high b values because they can be expected to rise faster than the market. Conversely, if markets are "bearish" and expected to fall, then securities with low beta factors are more attractive because they can be expected to fall less than prices overall.

To validate the CAPM, however, there are other assumptions (many of which should be familiar) that we will question later. For the moment, they are simply listed in Table 6.1 without comment to develop our analysis.

6.2 The Mathematical Derivation of the CAPM

Given the perfect market assumptions of the single period-index CAPM, consider an investor who initially places nearly all their funds in a portfolio reflecting the composition of the market. They subsequently invest the balance in security *j*. Using sequential numbering from previous equations, let us define R(P) the expected return on the revised portfolio as the weighted average of the expected returns of the individual components. This is given by adapting Equation (1) the basic formula for portfolio return from Chapter Two (remember?).

(37) $R(P) = x\, r_j + (1-x)\, r_m$

Where:

x = an extremely small proportion,
r_j = expected rate of return on security *j*,
r_m = expected rate of return on the market portfolio.

Subject to the original model's non-negativity constraints and requirements that sources of funds equal uses, the portfolio variance is also based on Equation (2) from Chapter Two:

(38) $VAR(P) = x^2\ VAR\ (r_j) + (1\text{-}x)^2\ VAR(r_m) + 2x\ (1\text{-}x)\ COV(r_j,r_m)$

The portfolio will be efficient if it has the lowest degree of risk for the highest expected return, given by the objective functions:

MAX: R(P), given VAR(P)

MIN: VAR(P), given R(P)

But note what has happened. By introducing security *j* into the market portfolio, the investor has altered the risk-return characteristics of their original portfolio. According to Sharpe and others, the *marginal return per unit of risk* is derived by:

1) *Differentiating* R(P) with respect to the investment in security *j*; $\Delta\ R(P)/\ \Delta x$,
2) *Differentiating* VAR(P) with respect to the investment in security *j*; $\Delta VAR(P)/\ \Delta\ x$.
3) *Solving* $\dfrac{\Delta\ R(P)/\ \Delta x}{\Delta\ VAR(P)/\ \Delta x}$ as $x \to 0$

Since (iii) above simplifies to $\Delta R(P)/\ \Delta VAR(P)$ as x tends to zero, *the incremental return per unit of risk* is therefore given by:

$$(39) \qquad \frac{\Delta R(P)}{\Delta VAR(P)} = \frac{r_m - r_j}{2(1-\beta_j)\ VAR\ (r_m)} \qquad \text{for } x \to 0$$

However, you will recall from our explanation of the SML that an investor can either borrow or lend at the risk-free rate of interest (r_f) with a beta value of zero. So, by incorporating a risk-free investment or a liability (if x is negative) the incremental rate of return given by Equation (39) is established by substituting rj = r_f and $\beta_j = 0$ into the equation such that:

$$(40) \qquad \frac{\Delta R(P)}{\Delta VAR(P)} = \frac{r_m - r_f}{2VAR\ (r_m)}$$

In a perfectly competitive capital market, the *incremental* risk-return trade-off must be the same for all investors. So, Equations (39) and (40) are identical:

$$(41) \quad \frac{r_m - r_j}{2(1-\beta_j)\,VAR(r_m)} \quad \text{is equivalent to} \quad \frac{r_m - r_f}{2\,VAR(r_m)}$$

Now, multiplying both sides of Equation (41) by the denominator on the left hand side and rearranging terms, Sharpe's *one* period, *single* factor Capital Asset Pricing Model (CAPM) for individual investments (explained earlier) is confirmed as follows:

(34) $r_j = r_f + (r_m - r_f)\ \beta_j$

And because systematic betas apply to a *portfolio*, as well as an *individual* investment we can define R(P) using our earlier notation

(35) $r_p = r_f + (r_m - r_f)\ \beta_p$

Remember, the CAPM is a *one period* model because the independent variables, r_f, r_m and b_j are assumed to remain constant over the time horizon. It is also a *single factor* model because systematic risk is prescribed entirely by the beta factor.

Equation (34) represents the expected rate of return on security *j*, which comprises a risk free return plus a premium for accepting market risk (the market rate minus the risk free rate), assuming that all correctly priced securities will lie on the SML. The market portfolio offers a premium $(r_m - r_f)\ \beta_j$ over the risk-free rate, r_f, which may differ from the *j*th security's risk premium measured by the beta factor β_j.

Thus, Sharpe's CAPM (like the others mentioned earlier, Lintner *et. al.*) enables an investor to establish whether individual securities (or portfolios) are under or over-priced, since the linear relationship between their expected rates of return and beta factors (systematic risk) can be compared with the SML (the market index).

6.3 The Relationship between the CAPM and SML

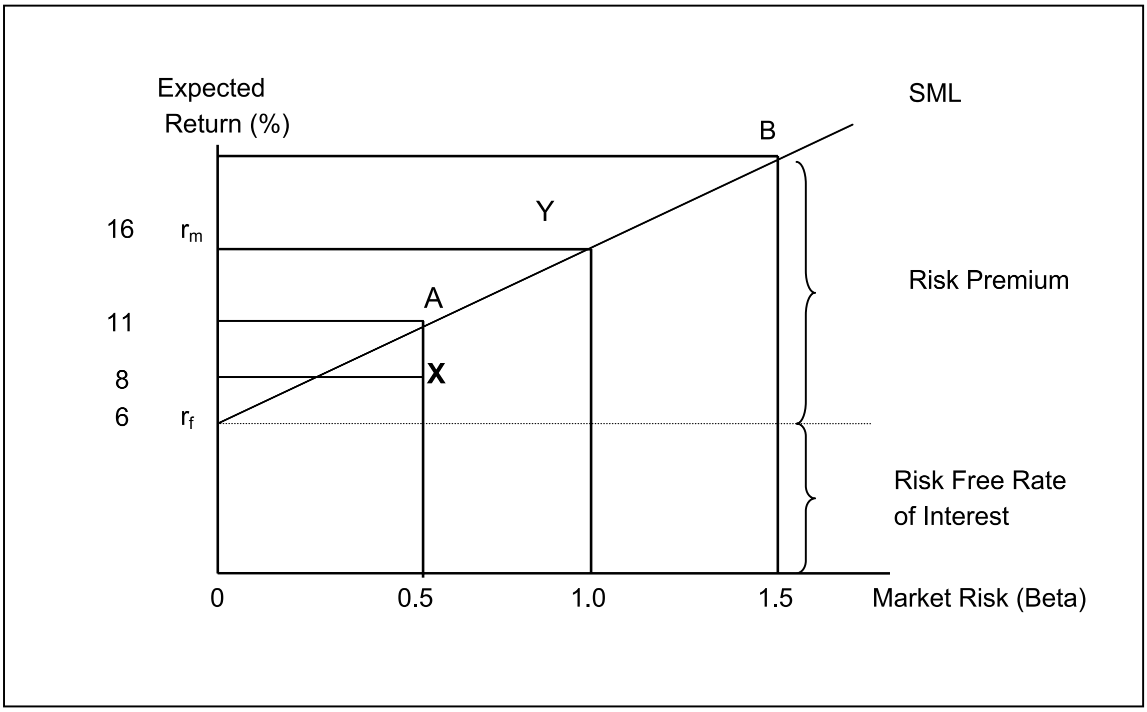

Figure 6.1: The CAPM and SML

Activity 1

Take a look at Figure 6.1. This is a reproduction of Figure 5.3, the Security Market Line (SML) explained in Chapter Five. At one extreme we have the expected return on risk-free investment (r_f) with a beta value of zero. At the other, portfolio B is a *borrowing* or *leveraged* portfolio with a beta of 1.5, which contains securities purchased by borrowing at the risk-free rate of interest. However, superimposed on the new graph are other beta values associated with expected returns, one of which is defined by the point X.

Explain its portfolio implications for rational, risk-averse investors.

Suppose we are considering investing in the security denoted by X on the graph with an expected return of 8 per cent and a beta coefficient of 0.5. We can see that the return is too low for the risk involved and that the security is overpriced because X is located below the SML. Consequently, rational investors wishing to sell their holdings would need to drop their price and increase the return (yield) until it impinges upon the SML at point A.

Given the slope of the SML defined by a risk free rate of 6 per cent and a market return of 16 per cent from a risky balanced portfolio, Figure 6.1 illustrates why the new *equilibrium* rate of return A with a beta value of 0.5 should be 11%. You can confirm this using the CAPM model:

(34) $r_j = r_f + (r_m - r_f)\ \beta_j$

where the expected return equals the risk-free rate, plus the market rate minus the risk-free rate, multiplied by the beta factor.

$\underline{11\%} = 6\% + (16\% - 6\%).0.5$

It is also clear from Figure 6.1 why investing in a security such as Y is beneficial. Stocks above the line will be in great demand, so they will rise in price causing a fall in yield.

From our examination of the data we can therefore draw the following conclusions.

In theoretical efficient capital markets in equilibrium that assimilate all information concerning a security into its price, all securities (or portfolios) will lie on the SML.

Individual investors need not conform to the market portfolio. They need only determine how much systematic risk they wish to assume, leaving market forces to ensure that any security can be expected to yield the appropriate return for its beta.

6.4 Criticism of the CAPM

Like much else in modern financial theory, critics of the CAPM maintain that its assumptions are so restrictive as to invalidate its conclusions, notably investor rationality, perfect markets and linearity. Moreover, the CAPM is only a single-period model, based on estimates for the risk-free rate, market return and beta factor, which are all said to be difficult to determine in practice. Finally, the CAPM also assumes that investors will hold a well diversified portfolio. It therefore ignores unsystematic risk, which may be of vital importance to investors who do not. However, as we have emphasised elsewhere in our studies, the relevant question is whether a model works, despite its limitations?

Although there is evidence by Black (1993) to suggest that the CAPM does not work accurately for investments with very high or low betas, overstating the required return for the former and understating the required return for the latter (suggesting compensation for unsystematic risk) most tests validate the CAPM for a broad spectrum of beta values.

The beta-return characteristics of individual securities also hold for portfolios. In fact, the beta of a portfolio seems more stable because fluctuations among its constituents tend to cancel each other out.

Way back in 1972, Black, Jensen and Scholes analysed the New York Stock Exchange over a 35 year period by dividing the listing into 10 portfolios, the first comprising constituents with the lowest beta factors and so on. Based on time series tests and cross-sectional analyses they found that the intercept term was not equal to the risk-free rate of interest, r_f, (which they approximated by 30 day Treasury bills). However, their study revealed an almost *straight-line* relationship between a portfolio's beta and its average return.

Critics still maintained that beta will only be stable if a company's systematic risk remains the same because it continues in the same line of business. However, subsequent studies using historical data to establish the stability of beta over time confirmed that if beta factors are calculated from past observable returns this problem can be resolved.

- The longer the period analysed, the better.
- The more data, the better, which suggests the use of a *sector* beta, rather than a *company* beta.

As an alternative to the basic CAPM, Black (1972) also tested a *two-factor* model, which assumed that investors couldn't borrow at a risk free rate but at a rate, r_z, defined as the return on a portfolio with a beta value of zero. This is equivalent to a portfolio whose covariance with the market portfolio's rate of return is zero.

(42) $r_j = r_z + (r_m - r_z)\ \beta_j$

The Black two-factor model confirmed the study by Black, Jensen and Scholes (*op.cit.*) and that a zero beta portfolio with an expected return, r_z exceeds the risk free rate of interests, r_f.

Despite further modifications to the original model, which need not detain us here, (multi-factors, multi-periods) the CAPM in its traditional guise continues to attract criticism, particularly in relation to its fundamental assumptions.

For example, even if we accept that all investors can borrow or lend at the risk-free rate, it does not follow that r_f describes a risk-free investment in *real* terms. Future inflation rates are neither pre-determined, nor affect individuals equally.

Marginal adjustments to a portfolio's constituents may also be prohibited by substantial transaction costs that outweigh their future benefits.

The fiscal system can also be *biased* with differential tax rates on income and capital gains. So much so, that different investors will construct or subscribe to portfolios that minimise their personal tax liability (a *clientele* effect).

And what if the stock market is *inefficient*? As we have discussed at great length in this study and elsewhere in our *SFM* companion texts, investors can not only profit from legitimate data by paying for the privilege. With access to insider information, which may even anticipate global events (such as the 1987 crash, millennium dot. com. fiasco and 2007 meltdown) perhaps they can also destabilise markets.

Conversely, even if we assume that the market is *efficient*, it has not always responded to significant changes in information, ranging from patterns of dividend distribution, takeover activity and government policies through to global geo-political events. Why else do even professional *active* managed portfolio funds periodically under-perform relative to the market index? The only way to "beat" the market, or so the argument goes, is either through pure speculation or insider information. Otherwise, adopt a *passive* policy of "buy and hold" to track the market portfolio and hope for the best

Other forces are also at work to invalidate the CAPM. You will recall that the model implies that the optimum portfolio is the market portfolio, which lies on the Security Market Line (SML) with a beta factor of one. Individual securities and portfolios with different levels of risk (betas) can be priced because their expected rate of return and beta can be compared with the SML. In equilibrium, all securities will lie on the line, because those above or below are either under or over priced in relation to their expected return. Thus, market demand, or the lack of it, will elicit either a rise or fall on price, until the return matches that of the market.

However, we have a problem, namely how to define the market. It is frequently forgotten that the CAPM is a *linear mode*l based on *partial equilibrium analysis* that subscribes to the Modigliani-Miller (MM) *law of one price*. Based on their arbitrage process, (1958 and 1961) explained in our *SFM* companion texts, you will recall that two similar assets must be valued equally. In other words, two portfolio constituents that contribute the same amount of risk to the overall portfolio are *close substitutes*. So, they should exhibit the same return. But what if an asset has no close substitute, such as the market itself? How do we establish whether the market is under or overvalued?

As Roll (1977) first noted, most CAPM tests may be invalid because all stock exchange indices are only a *partial* measure of the *true* global market portfolio. Explained simply, by definition the market portfolio should include every security world-wide.

To prove the point, Roll demonstrated that a change in the surrogate for the American stock market from the Standard and Poor 500 to the Wilshire 5000 could radically alter a security's expected return as predicted by the CAPM. Furthermore, if betas and returns derived from a stock market listing were unrelated, the securities might still be priced correctly relative to the global market portfolio. Conversely, even if the listing was efficient (shares with high betas did exhibit high returns) there is no obvious reason for assuming that each constituent's return is only affected by global systematic risk.

A further criticism of the CAPM is that however one defines the capital market, movements up and down are dominated by price changes in the securities of larger companies, Yet as Fama and French (1992) first observed, it is to these companies that institutional portfolio fund managers (active or passive) are attracted, though they may under-perform relative to smaller companies. Explained simply, fund managers with perhaps billions to spend are hostages to fortune, even in a "bull" scenario. They have neither the time, nor research budgets to scrutinise innumerable companies "neglected" by the market with small capitalisations based on little information.

Turning to "bear" markets characterised by rising systematic risk, multi-national portfolio fund managers still have little room for manoeuvre. According to Hill and Meredith (1994):

The first option is to liquidate all or part of a portfolio. However, if the whole portfolio were sold it could be difficult to dispose of a large fund quickly and efficiently without affecting the market. Unlike a private investor, total disposal may also be against the fund's trust deed. If only part of the portfolio was liquidated there is the further question of which securities to sell.

The second option is to reduce all holdings, to be followed by subsequent reinvestment when the market bottoms out. However, the fall in prices may have to be in excess of 2 per cent to cover transaction and commission costs,).

Clearly, both alternatives may be untenable and impose significant constraints upon the opportunities to control risk. Indeed, those sceptical of portfolio management generally and the CAPM in particular, regard successful investment as a matter of luck rather than judgement, insider information, or unlikely economic circumstances where all prices move in unison.

Review Activity

Assuming the risk-free rate and expected return on the market portfolio for Muse plc are 10 per cent and 18 per cent respectively:

(1) Use the CAPM to calculate the expected returns on stocks with the following beta values:

$$\beta = 0, 0.5, 1.0, 1.5$$

(2) How would each stock fit into the investment plans for an actively managed portfolio?

(1) Using the data and Equation (34) to derive the expected returns, the CAPM reveals that if:

$$\beta = 0, 0.5, 1.0 \text{ or } 1.5$$

$$r_j = 10 + (18 - 10)\beta = 10\%, 14\%, 18\% \text{ and } 22\%, \text{ respectively}$$

(2) The investment plans for an actively portfolio can be explained as follows.

With a beta value greater than one, a stock's expected return should "beat" the market and *vice versa.* A beta of one produces a return equal to the market return and a beta value of zero produces an expected return equal to the risk-free rate.

Thus, we can classify investment into three broad categories of risk for the purpose of "active" portfolio management:

$$\beta > 1.0 = \textit{Aggressive}$$
$$\beta < 1.0 = \textit{Defensive}$$
$$\beta = 1.0 = \textit{Neutral}$$

A portfolio manager's interest in each category of beta factor concerns the likely impact of changes in a market index on the share's expected return. Aggressive shares can be expected to outperform the market in either direction. If the return on the index is expected to rise, the returns on high beta shares will rise faster. Conversely, if the market is expected to fall, then their returns will fall faster. Defensive shares with beta values lower than one will obviously under-perform relative to the market in each direction. Neutral shares will tend to shadow it.

Hence, rather than adopt a passive policy of "buy and hold" by constructing a *tracker* fund representative of a stock market index, "active" portfolio managers will wish to pursue:

An *aggressive* investment strategy by moving into *high* beta shares when stock market returns are expected to rise (a bull market).

A *defensive* strategy based on *low* beta shares and even risk-free assets with zero betas, when the market is about to fall (a bear market).

6.4 Summary and Conclusions

If the capital market is so unpredictable that it is impossible for investors to beat it using the CAPM, it is important to remember that the operational usefulness of alternative mean-variance analyses and expected utility models explained at the very beginning of this text are also severely limited in their application. This is why the investment community turned to Markowitz portfolio theory and the Sharpe CAPM for inspiration. And why others refined these models into a coherent body of work now termed Modern Portfolio Theory (MPT) to facilitate the efficient diversification of investment.

Since the new millennium, despite the volatility of financial markets and their tendency to crash (or perhaps because of it) the portfolio objectives of investors remain the same:

> To eliminate unsystematic risk and to establish the optimum relationship between the systematic risk of a financial security, project, or portfolio, and their respective returns; a trade-off with which investors feels comfortable.

So to conclude our studies, what does the *single- period* model CAPM based on Markowitz efficiency contribute to Strategic Financial Management within the context of their *multi-period* investment, dividend and financing decisions, which previous models considered throughout this text and *SFM* have failed to deliver?

6.5 Selected References

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7 Capital Budgeting, Capital Structure Andthe Capm

Introduction

So far, our study of Markowitz efficiency, beta factors and the CAPM has concentrated on the stock market's analyses of security prices and expected returns by financial institutions and private individuals. This is logical because it reflects the rationale behind the chronological development of Modern Portfolio Theory (MPT). But what about the impact of MPT on individual companies and their appraisal of capital projects upon which all investors absolutely depend? If management wish to maximise shareholder wealth, then surely a new project's expected return and systematic risk relative to the company's existing investment portfolio and stock market behaviour, like that for any financial security, is a vitally important consideration.

In this Chapter we shall explore the corporate applications of the CAPM by strategic financial management, namely:

- The derivation of a discount rate for the appraisal of capital investment projects on the basis of their systematic risk.
- How the CAPM can be used to match discount rates to the systematic risk of projects that differ from the current business risk of a firm.

Because the model can be applied to projects financed by debt as well as equity, we shall also establish a mathematical connection between the CAPM and the Modigliani-Miller (MM) theory of capital gearing based on their "law of one price" covered in our SFM companion texts.

7.1 Capital Budgeting and the CAPM

As an alternative to calculating a firm's weighted average cost of capital (WACC) explained in the *SFM* texts, the theoretical derivation of a project discount rate using the CAPM and its application to NPV maximisation is quite straightforward. A risk-adjusted discount rate for the *j*th project is simply the risk-free rate added to the product of the market premium and the *project* beta, given by the following expression for the familiar CAPM equation:

(45) $r_j = r_f + (r_m - r_f)\ \beta_j$

The project beta (β_j) measures the *systematic* risk of a specific project (more of which later). For the moment, suffice it to say that in many textbooks the project beta is also termed an *asset* beta denoted by β_A.

We then derive the expected NPV by discounting the average net annual cash flows at the risk-adjusted rate from which the initial cost of the investment is subtracted, using a mathematical formulation that you first encountered in Part Two of the *SFM* texts.

$$(46)\quad NPV = \sum_{t=1}^{n} C_t / (1+r_j)^t - I_0$$

Individual projects are acceptable if:

$$NPV \geq 0$$

Collectively, projects that satisfy this criterion can also be ranked for selection according to the size of their NPV. Given:

$NPV_A > NPV_B > \ldots NPV_N$ we prefer project A.

So far, so good; but remember that CAPM project discount rates are still based on a number of simplifying assumptions. Apart from adhering to the traditional concept of perfect capital markets (Fisher's Separation Theorem) and mean-variance analysis (Markowitz efficiency) the CAPM is only a *single-period* model, whereas most projects are *multi-period* problems.

According to the CAPM, all investors face the same set of investment opportunities, have the same expectations about the future and make decisions within one time horizon. Any new investment made *now* will be realised *then*, next year (say) and a new decision made.

Given the assumptions of perfect markets characterised by random cash flow distributions, there is no theoretical objection to using a *single-period* model to generate an NPV discount rate for the evaluation of a firm's *multi-period* investment plans. The only constraints are that the risk-free rate of interest, the average market rate of return and the beta factor associated with a particular investment are *constant* throughout its life.

Unfortunately, in reality the risk-free rate, the market rate and beta are rarely constant. However the problem is not insoluble. We just substitute *periodic* risk-adjusted discount rates (now dated $r_{j\,t}$) for a constant r_j into Equation (46) for each future "state of the world", even if only one of the variables in Equation (45) changes. It should also be noted that the phenomenon of multiple discount rates combined with different economic circumstances is not unique to the CAPM. As we first observed in Part Two of *SFM*, it is common throughout NPV analyses.

On first acquaintance, it would therefore appear that the application of a CAPM return to capital budgeting decisions provides corporate financial management with a practical alternative to the WACC approach. A particular weakness of WACC is that it defines a single discount rate applicable to *all* projects, based on the assumptions that their acceptance doesn't change the company's risk or capital structure and is *marginal* to existing activities. In contrast, the CAPM rate varies from project to project, according to the systematic risk of each investment proposal. However, the CAPM still poses a number of problems that must be resolved if it is to be applied successfully, notably how to derive an appropriate *project* beta factor and how to measure the impact of *capital gearing* on its calculation.

For these reasons, we shall defer a comprehensive numerical example of investment appraisal and the CAPM until you read the Exercises associated with this chapter, by which time we will have covered the issues involved.

7.2 The Estimation of Project Betas

For simplicity throughout previous chapters we have used a *general* beta factor (β) applicable to the *overall* systemic risk of portfolios, securities and projects. But now our analysis is becoming more focussed, *precise* notation and definitions are necessary to *discriminate* between systemic *business* and *financial* risk. Table 7.1 summarises the beta measures that we shall be using for future reference and also highlights a number of problems.

β = total *systematic* risk, which relates portfolio, security and project risk to *market* risk.
β_j = the *business* risk of a specific project (*project* risk) for investment appraisal.
β_E = the *published* equity beta for a company that incorporates business risk and systematic *financial* risk if the firm is geared.
β_A = the overall business risk of a firm's *assets* (projects). It also equals a company's *deleveraged* published beta (β_E) which measures business risk *free* from financial risk.
β_D = the beta value of debt (which obviously equals zero if it is risk-free).
β_{EU} and β_{EG} are the respective equity betas for similar all-share and geared companies.

Table 7.1: Beta Factor Definitions

When an all-equity company is considering a new project with the same level of risk as its current portfolio of investments, total systematic risk *equals* business risk, such that:

$$\beta = \beta_j = \beta_E = \beta_A = \beta_{EU}$$

When a company is funded by a combination of debt and equity, this series of equalities must be modified to incorporate a *premium* for systematic *financial* risk. As we shall discover, the equity beta (β_E) will be a *geared* beta reflecting business risk *plus* financial risk, which measures shareholder exposure to debt in their firm's capital structure. Thus, the equity beta of an all-share company is always lower than that for a geared firm with the same business risk.

$$\beta_{EU} < \beta_{EG}$$

Irrespective of a gearing problem, Table 7.1 reveals a further weakness of the CAPM. A company's asset beta (β_A) should produce a discount rate that is appropriate for evaluating projects with the same overall risk as the company itself. But what if a new project does not reflect the average risk of the company's assets? Then the use of β_A is no more likely to produce a correct investment decision than the use of a WACC calculation.

To illustrate the point, Figure 7.1 graphs the Security Market Line (SML) to show the required return on a project for different beta factors, with a company's WACC. The use of the overall cost of capital to evaluate projects whose risk differs from the company's average will be sub-optimal where the IRR of the project is in either of the two shaded sections. To calculate the correct CAPM discount rate using Equation (45), we must determine the project beta.

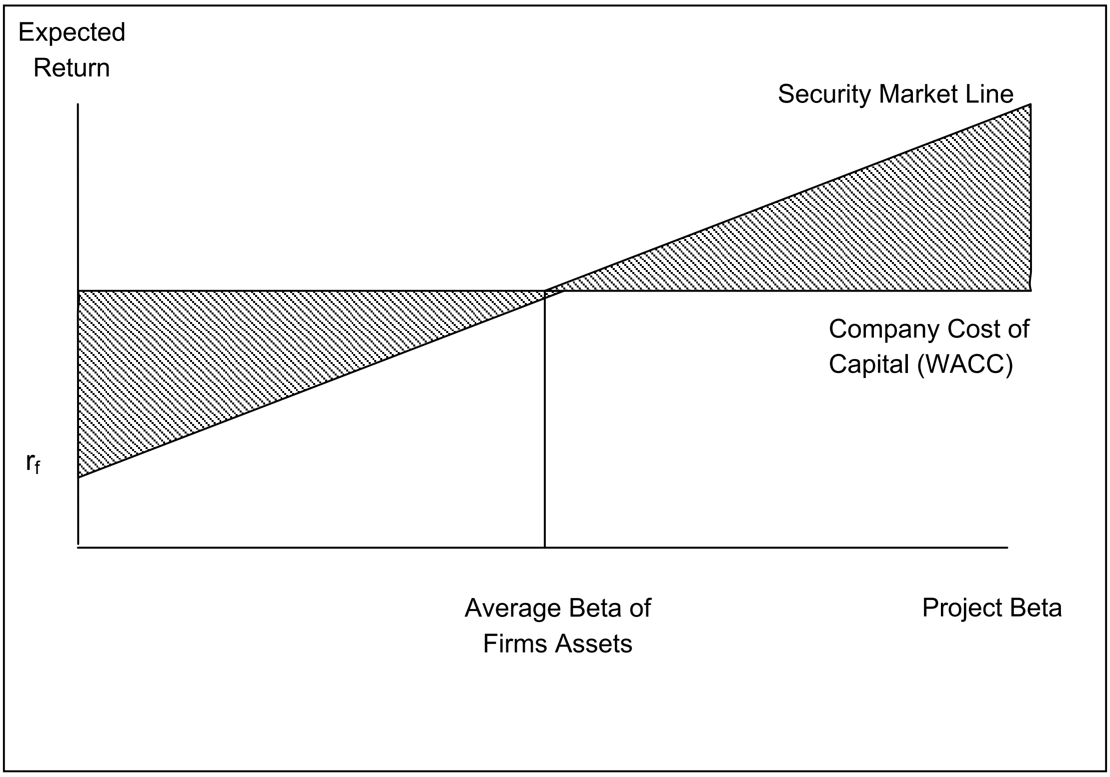

Figure 7.1: The SML, WACC and Project Betas

The company's average beta, shown in the diagram, provides a measure of risk for the firm's overall returns compared with that of the *market*. However, management's investment decision is whether or not to invest in a *project*. So, like the WACC, if the project involves diversification away from the firm's core activities, we must use a beta coefficient appropriate to that class of investment. The situation is similar to a stock market investor considering whether to purchase the shares of the *company*. The individual would need to evaluate the share's return by using the *market* beta in the CAPM.

Even if diversification is not contemplated, the project's beta factor may not conform to the *average* for the firm's assets. For example, the investment proposal may exhibit high *operational gearing (*the proportion of fixed to variable costs) in which case the project's beta will exceed the average for existing operations.

A serious conflict (the *agency* problem) can also arise for those companies producing few products, or worse still a single product, particularly if management approach their capital budgeting decisions based on self-interest and short-termism, rather than shareholder preferences. Shareholders with well-diversified corporate holdings who dominate such companies may prefer to see projects with high risk (high beta coefficients) to balance their own portfolios. Such a strategy may carry the very real threat of bankruptcy but in the event may have very little impact their overall returns. For corporate management, the firm's employees and its suppliers, however, the policy may be economic suicide.

Fortunately, if a beta is required to validate the CAPM for project appraisal, help is at hand. Management can obtain factors for companies operating in similar areas to the proposed project by subscribing to the many commercial services that regularly publish beta coefficients for a large number of companies, world wide. Their listings also include stock exchange classifications for *industry* betas. These are calculated by taking the market average for quoted companies in the same industry. Research reveals that the measurement errors of individual betas cancel out when industry betas are used. Moreover, the larger the number of comparable beta constituents, the more reliable the industry factor.

> So, if management wish to obtain an estimate for a project's beta, it can identify the industry in which the project falls, and use that industry's beta as the project's beta. This approach is particularly suitable for highly *diversified* and *divisionalised* companies because their WACC or market beta would be of little relevance as a discount rate for its divisional operations.

As an alternative to stock market data, management can also estimate a project's beta from first principles by calculating its *F-value*.

> The F-value of a project is rather like a beta factor in that it measures the variability of a project's performance, *relative* to the performance of an entity for which a beta value exists.
>
> The entity could be the industry in which the project falls, the firm undertaking the project, or a division within the firm that is responsible for the project.

A project's F-value is defined as follows:

(47) $$F = \frac{\text{Percentage change in the project's performance}}{\text{Percentage change in the "entity's" performance}}$$

As a result, we can obtain an estimate of a project's beta through one of three routes:

(i) $$\frac{\text{\% change in the company's performance}}{\text{\% change in the industry's performance}} \times \beta_{\text{industry}}$$

(ii) $$\frac{\text{\% change in the project's performance}}{\text{\% change in the company's performance}} \times \beta_{\text{company}}$$

(iii) $$\frac{\text{\% change in the project's performance}}{\text{\% change in the division's performance}} \times \beta_{\text{division}}$$

$\Bigg\}\ \beta_{\text{project}}$

Activity 1

Let us suppose that a company's divisional management is considering a capital project, whose performance may be affected 15 per cent either way, depending on whether the division's overall performance rises or falls by 10 per cent. In other words, the project's profitability is expected to be more volatile than that of the division because of specific economic factors.

Calculate the project's F-value and estimate the project's beta coefficient given the division's beta factor is 0.80.

Using Equation (47) we can calculate the F-value as follows:

$$F = 15\% / 10\% = \underline{1.5}$$

If the divisional beta value is 0.80, then the project beta (β_{project}) can be estimated as follows:

(% change in the project's performance / % change in the division's performance) $\times \beta_{\text{division}}$

$$\beta_{\text{project}} = 1.5 \times 0.80 = \underline{1.2}$$

7.3 Capital Gearing and the Beta Factor

The CAPM defines an individual investment's risk relative to a well-diversified portfolio as *systematic risk.* Measured by the beta coefficient, it is the only risk a company or an investor will pay a premium to avoid. You will recall from Chapter Four (Figure 4.3) that it can be sub-divided into:

- *Business risk* that arises from the variability of a firm's earnings caused by market forces,
- *Financial risk* associated with dividend policies and capital gearing, both of which may amplify business risk

Without getting enmeshed in dividend policies, we shall accept the 1961 MM hypothesis that they are *irrelevant*. Based on their "law of one price" (covered in the *SFM* texts and for which there is considerable empirical support) *financial* risk should not matter in an all-equity company. Applied to the CAPM, the *systematic* risk of investors (who are all shareholders) can be defined by the *business* risk of the firm's underlying asset investments.

The *equity* beta of an unlevered (all-equity) firm equals an *asset* beta, which measures the business risk of all its investments relative to the market for ordinary shares (common stock). Using earlier notation:

$$\beta_{E\,U} = \beta_A$$

The CAPM return on project (r_j) is then defined by:

(48) $r_j = r_f + (r_m - r_f)\ \beta_A$

If there is no debt in the firm's capital structure, the company's asset (equity) beta equals the *weighted average* of its individual project betas (b_i) based on the market value of equity.

(49) $\beta_A = \Sigma\ w_i\ \beta_i = \beta_{EU}$

But what about companies who decide to fund future investments by gearing up, or the vast majority who already employ debt finance?

To make rational decisions, it would appear that management now require an asset beta to measure a firm's business risk that an ungeared equity beta can no longer provide. For example, an all-equity company may be considering a take-over that will be financed entirely by debt. To assess the acquisition's viability, management will now need to calculate their overall CAPM return on investment using an asset beta that reflects a *leveraged* financial mix of fixed interest on debt and dividends on shares.

Later in this chapter we shall resolve the dilemma using the predictions of MM's capital structure hypothesis (1958). Based on their law of one price, whereby similar firms with the same risk characteristics (except capital gearing) cannot sell at different prices, it confirms their dividend hypothesis, namely that financial policy is irrelevant. First, however, let us develop the CAPM, to illustrate the relationship between an asset beta and the equity and debt coefficients for a geared company.

When a firm is financed by a debt-equity mix, its earnings stream and associated risk is divided between the firm's shareholders and providers of corporate debt. The proportion of risk reflects the market values of debt and equity respectively, defined by the *debt-equity ratio*. So, the equity beta will be a *geared* equity beta. It not only incorporates business risk. It also determines shareholders' exposure to financial risk defined by the proportion of contractual, fixed interest securities in the capital structure. For this reason the equity beta of an unlevered company is always lower than the beta of a levered company.

Given a geared equity beta (β_E) and debt beta (β_D), the asset beta (β_A) for a company's investment in risky capital projects can be expressed as a weighted average of the two:

(50) $$\beta_A = \beta_{E\,G}\,[V_E / (V_E + V_D)] + \beta_D\,[V_D / (V_E + V_D)]$$

Where:

V_E and V_D are the *market* values of equity and debt, respectively,

V_E *plus* V_D define the firm's total market value (V).

Activity 2

A firm with respective market values of €60m and €30m for equity and debt has an equity beta of 1.5. The debt beta is zero.

(1) Use Equation (50) to calculate the asset beta (β_A).

(2) Explain a simplified mathematical structure of the calculation.

(1) The asset beta (β_A) calculation

(50) $\beta_A = \beta_{EG}\,[V_E / (V_E + V_D)] + \beta_D\,[V_D / (V_E + V_D)]$
$= 1.5\,[60/(60+30)] + 0\,[30/(60+30)] = \underline{1.0}$

(2) The mathematical structure of β_A.

When a company is financed by debt and equity, management need to derive an asset beta using the *weighted average* of its geared equity and debt components. The market values of debt and equity provide the weightings for the calculation. Note, however, that because the market risk of debt (β_D) was set to zero, the right hand side of Equation (50) disappears.

This is not unusual. As explained in *SFM*, debt has priority over equity's share of profits and the sale of assets in the event of liquidation. Thus, debt is more secure and if it is risk-free, there is no variance. So if β_D equals zero, our previous equation for an asset beta reduces to:

(51) $\beta_A = \beta_{EG}\,[V_E / (V_E + V_D)]$

For example, if a company has an equity beta of 1.20, a debt-equity ratio of 40 per cent and we assume that debt is risk-free, the asset beta is given by:

$\beta_A = 1.20\,[100 / (100 + 40)]$
$= \underline{0.86}$

Perhaps you also recall from *SFM* that debt is also a *tax deductible* expense in many economies. If we incorporate this fiscal adjustment into the previous equations (where t is the tax rate) we can redefine the mathematical relationship between the asset beta and its geared equity and debt counterparts as follows.

(52) $\beta_A = \beta_{E\,G} \{V_E / [V_E + V_D(1\text{-}t)]\} + \beta_D \{[V_D(1\text{-}t) / (V_E + V_D(1\text{-}t))]\}$

(53) $\beta_A = \beta_{E\,G} \{V_E / [V_E + V_D(1\text{-}t)]\}$ if debt is risk-free

Despite the tax effect, our methodology for deriving a company's asset beta still reveals a *universal* feature of the CAPM that financial management can usefully adopt to assess individual projects.

Whenever risky investments are combined, the asset beta of the resultant portfolio is a *weighted average* of the debt and equity betas.

Activity 3

Consider a company with a current asset beta of 0.90. Itaccepts a project with a beta of 0.5 that is equivalent to 10 per cent of its corporate value after acceptance.

Confirm that:

1. The new (*ex-post*)beta coefficient of the company equals 0.86.
2. The *new project* reduces the original (*ex-ante*) risk of the firm's *existing portfolio.*

7.4 Capital Gearing and the CAPM

The CAPM defines a project's discount rate as a return equal to the risk-free rate of interest, plus the product of the market premium and the project's asset beta (a risk premium) to compensate for systematic (business) risk. However, we now know that the financial risk associated with capital gearing can also affect beta factors. So, the discount rate derived from the CAPM for investment appraisal must also be affected, but how?

Let us first consider a company funded entirely by equity that is considering a new project with the same level of risk as its existing activities. The firm's equity beta (β_{EU}) can be used as the project's asset beta (β_A) because the shareholders' return (K_e) equals the company's return (r_j) on a new project of equivalent risk. So, the project return that provides adequate compensation for holding shares in the company is the equity return (K_e) obtained by substituting the appropriate equity beta (β_E) into the familiar CAPM formula.

(54) $K_e = r_j = r_f + (r_m – r_f) \beta_{EU}$

The CAPM therefore offers management an important alternative to the derivation of project discount rates that use the traditional dividend and earnings valuation models explained in the *SFM* texts. In an *unlevered* (all-equity) firm, the *shareholders'* return (K_e) defines the *company's* cost of capital (K_U) as follows:

(55) $K_U = K_e = r_j = r_f + (r_m - r_f) \beta_{EU}$

The question we must now ask is whether Equation (55) has any parallel if the firm is geared.

The short answer is yes. Rather than use traditional dividend, earnings and interest models to derive a WACC (explained in *SFM*) we can substitute an appropriately *geared* asset beta for an *all-equity* beta into the CAPM to estimate the overall return on debt and equity capital for project appraisal.

(56) $K_G = r_j = r_f + (r_m - r_f) \delta_A$

7.5 Modigliani-Miller and the CAPM

Without debt in it capital structure, a company's asset beta equals its equity beta for projects of equivalent risk. However, according to MM's theory of capital structure (*op. cit.*) based on their "law of one price" and the *arbitrage* process, companies that are identical in every respect apart from their gearing should also have the same asset betas. Because their business risk is the same, the factors are not influenced by methods of financing. To summarise MM's position

An ungeared company's asset beta equals its equity beta.
A geared company's asset beta is lower than its equity beta.
Irrespective of gearing, the asset beta for any company equals the equity beta of an ungeared company with the same business risk.
The asset beta (equity beta) of an unlevered company can be used to evaluate projects in the same risk class without considering their finance.

$$\beta_j = \beta_A = \beta_{EU} < \beta_{EG}$$

You will recall from your studies that MM's capital theory (like their dividend irrelevancy hypothesis) depends on perfect market assumptions. However, because these assumptions also underpin much else in finance (including the CAPM) for the moment we shall accept them. To illustrate the MM relationship between the beta factors of all-equity and geared companies with the same systemic business risk, let us begin with the following equation using our familiar notation in a taxless world.

(57) $\beta_A = \beta_{EU} = \beta_{EG} [V_E / (V_E + V_D)] + \beta_D [V_D / (V_E + V_D)]$

If we now rearrange terms, divide through by V_E and solve for β_{EG}, the mathematical relationship between the geared and ungeared equity betas can be expressed as follows:

(58) $\beta_{EG} = \beta_{EU} + (\beta_{EU} - \beta_D) V_D / V_E$

This equation reveals that the equity beta in a geared company equals the equity beta for an all-share company in the same class of business risk, *plus* a premium for systemic financial risk. The premium represents the difference between the all-equity beta and debt beta multiplied by the debt-equity ratio. However, the important point is that the increase in the equity beta measured by the risk premium is exactly offset by a lower debt factor as the firm gears up leaving the asset beta unaffected. In other words, irrespective of leverage, the asset betas of the two firms are still identical and equal the equity beta of the ungeared firm.

$$\beta_A = \beta_{EU} < \beta_{EG}$$

For those of you familiar with MM's capital structure hypothesis, the parallels are striking. According to MM, the expected return on equity for a geared firm (K_{eG}) relative to the return (K_{eU}) for an all-share firm in a taxless world equals:

$$(59)\ K_{eG} = K_{eU} + (K_{eU} - K_d)\ V_D / V_E.$$

This states that the return for a geared firm equals an all-equity return for the same class of business risk, *plus* a financial risk premium defined by the difference between the all-equity return and the cost of debt multiplied by the debt-equity ratio. The premium compensates shareholders for increasing exposure to financial risk as a firm gears up. As we observed in *SFM*, however, because the cheaper cost of debt exactly offsets rising equity yields, the overall cost of capital (WACC) is unaffected. So, irrespective of leverage, all firms with the same business risk can use the cost of equity for an all-share firm as a project discount rate before considering methods of financing.

Turning to a world of taxation, where debt is a *tax-deductible* expense with a tax rate (t), we can redefine the equity beta of a geared company from Equation (58) as follows:

$$(60)\ \beta_{EG} = \beta_{EU} + [(\beta_{EU} - \beta_D)\ (1\text{-}t)\ V_D / V_E]$$

And if debt is risk-free with *zero* variance, so that β_D is zero, the formula simplifies to:

$$(61)\ \beta_{EG} = \beta_{EU} + [(\beta_{EU}\ (1\text{-}t)\ V_D / V_E]$$

Review Activity

To illustrate the union between MM and the CAPM, consider a leveraged company in an economy where interest is tax deductible at a 20 per cent corporate rate. 20 million ordinary shares are authorised and issued at a current market value of £2.00 each (*ex-div*). The equity beta is 1.5. Debt capital comprises £10 million, irredeemable 10 per cent loan stock, currently trading at par value.

Calculate the company's asset beta and briefly explain the result.

Since the equity beta for an *ungeared* company equals the asset beta for any company in the same risk class, we can use Equation (61) to solve for β_{EU} and hence β_A as follows.

First, define the market values of equity and debt:

V_E = £2.00 × 20 million = £40 million
V_D = £10 million

Next, define the geared equity beta of 1.5 assuming that debt sold at par is risk-free ($\beta_D = 0$).

$$\beta_{EG} = 1.5 = \beta_{EU} + [\beta_{EU} (1\text{-}0.2)\ (10/40)]$$
$$= \beta_{EU} \{1 + [(1\text{-}0.2)\ (10/40)]\}$$

Finally, rearrange terms to solve for β_{EU} and β_A.

$$\beta_A = \beta_{EU} = 1.5/1.2 = \underline{1.25}$$

The result is to be expected. The asset beta should be smaller than the geared equity beta (i.e. 1.25 < 1.5) since the systemic risk associated with the asset investment is only one component of the total risk associated with the shares. The asset beta measures business risk, whereas the geared beta measures business and financial risk

7.5 Summary and Conclusions

If management use the CAPM rather than a WACC to obtain a risk-adjusted discount rate for project appraisal, they need to resolve the following questions

Question:	**Is the business risk of a project equivalent to that for the company?**	
Answer:	YES	NO
Solution:	Use the company's current equity beta	Use an equity beta for similar companies with similar projects

Question:	**Is the chosen equity beta affected by capital gearing?**	
Answer:	YES	NO
Solution:	De-leverage "ungear" the equity beta to derive an asset beta	Use an equity beta equivalent to an asset beta if it is not affected by gearing

Having obtained an appropriate asset beta, the project discount rate may then be calculated using the CAPM formula.

(62) $r_j = r_f + (r_m - r_f) b_A$

> According to MM's capital structure theory, the asset betas of companies, or projects, in the same class of business risk are identical irrespective of leverage. Higher equity betas are offset by lower debt betas, just as higher equity yields offset cheaper financing, as a firm gears up

Even in a taxed world, it is possible to establish a connection between MM and the CAPM. With tax, the MM cost of equity for a geared firm is given by:

(63) $K_{eG} = K_{eU} + [(K_{eU} - K_d) (1\text{-}t) V_D / V_E]$

According to the CAPM, the equity costs for an ungeared and geared firm are given by:

(64) $K_{eU} = r_j = r_f + (r_m - r_f) b_{EU}$

(65) $K_{eG} = r_j = r_f + (r_m - r_f) b_{EG}$

Where:

$b_A = b_{EU} < b_{EG}$

If we assume that the company's pre-tax cost of debt (K_d) in Equation (63) equals the risk-free rate (r_f) in Equations (64) and (65) we can write r_f for K_d in Equation (63). If we now substitute Equations (64) and (65) into Equation (63) rearrange terms and simplify the result, we can confirm our earlier equation for a geared equity beta:

$$(61)\ b_{EG} = b_{EU} + [b_{EU}(1\text{-}t)\ V_D / V_E]$$

$$= b_{EU}\{1 + [(1\text{-}t)\ (10/40)]\}$$

For an application of this formula and the derivation of the cost of equity using the CAPM see Exercise 7.2 in the companion text.

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Part IV: Modern Portfolio Theory

8 Arbitrage Pricing Theory and Beyond

Introduction

Previous chapters have presented a series of mathematical models representing a body of work termed Modern Portfolio Theory (MPT) available to financial management when making strategic investment decisions. MPT was originally developed for use by investors in securities, primarily fund managers and professional analysts with the time, resources and expertise to implement the models and interpret their findings. Today, anybody with access to a computer, the appropriate software and a reasonable financial education can *model* quite complex tasks. Ultimately, however, it is people who should *interpret* the results and not the computer. One lesson to be learnt from the1987 stock market crash is the catastrophic effect of automated trading. Another from the 2007 meltdown and ongoing financial crises is that computer driven models can be so complex that hardly anybody understands what is going on anymore.

Like all financial theories, MPT should therefore be a guide to human action and not a substitute. And while the benefits of IT cannot be overstressed, you should always understand the financial model that underpins the computer program you are running. So, let us review the original purpose of MPT, notably the CAPM and then outline its subsequent development, notably *Arbitrage Pricing Theory* (APT).

8.1 Portfolio Theory and the CAPM

You will recall that portfolio theory was initially developed by Harry Markowitz in the early 1950s to explain how rational investors in perfect markets can minimise the risk of investment without comprising return by diversifying and building up an efficient portfolio of investments. The risk of each portfolio is measured by the variability of possible returns about the mean measured by the standard deviation. Investor risk-return attitudes can be expressed by indifference curves.

In 1958, John Tobin explained how the introduction of risk-free investments into Markowitz' theory further reduces the risk of a portfolio. According to Tobin, the Capital Market Line (CML) defines a new "efficient frontier" of investments for all investors.

Applied to project appraisal, Markowitz theory reveals that an individual project's risk is not as important as its effect on the portfolio's overall risk. So, whenever management evaluate a risky project they must correlate the individual project risk with that for the existing portfolio it will join to assess its suitability.

Without the benefit of today's computer technology, the mathematical complexity of the Markowitz model arising from its covariance calculations prompted other theorists to develop alternative approaches to efficient portfolio diversification. In the early 1960s by common consensus, the CAPM emerged as a means whereby investors in financial securities were able to reduce their total risk by constructing portfolios that discriminate between systematic (market risk) and unsystematic (specific) risk.

The CAPM (usually associated with its prime advocate William Sharpe) states that the return on a security or portfolio depends on whether their prices follow prices in the market as a whole by reference to a suitable index, such as the FT-SE 100. The closer the correlation between the price of either an individual security or a portfolio and this market proxy (measured by the beta factor) the greater will be their expected returns. Thus, if an investor knows the beta factor (relative risk) of a security or portfolio, their returns can be predicted with accuracy. Profitable trading of portfolios is then accomplished by buying (selling) undervalued (overvalued) securities relative to their systematic or market risk.

The CAPM also states that rational investors would choose to hold a portfolio that comprises the stock market as a whole. By definition, the market portfolio has a beta of one and is the most "efficient" in the sense that no other combination of securities would provide a higher return for the same risk. You will recall that it is a benchmark by which the CAPM establishes the Security Market Line (SML) in order to compare other beta factors and returns. From this linear relationship, rational investors can ascertain whether individual shares are underpriced or overpriced and determine other efficient portfolios that balance their personal preference for risk and return.

According to the CAPM:

Any security with the same risk as the market will have a beta of 1.0; half as risky it will have a beta of 0.5; twice as risky it will have a beta of two.

The required rate of return given by the CAPM formula is composed of the return on risk-free investments, *plus* a risk premium measured by the difference between the market return and the risk free rate multiplied by an appropriate beta factor. For example, using Equation (45) for an investment with a beta of b_j:

$$r_j = r_f + (r_m - r_f)\, b_j$$

If we use the CAPM for project appraisal, rather than stock market analysis, the procedure remains the same. Essentially, we are substituting an investment project for a security into a company's portfolio of investments, rather than a market portfolio. Risk relates to the cost of capital and management's objective is to obtain a discount rate to appraise individual projects.

8.2 Arbitrage Pricing Theory (APT)

So far, so good, but if we consider the purpose for which the CAPM was originally intended, namely stock market investment, it has limitations. As we observed in Chapter Six, even the most actively managed, institutional portfolio funds periodically underperform relative to the market as a whole.

Leaving aside the questionable assumptions that investors are rational, markets are efficient and prices perform a "random walk" (dealt with in our Introduction and elsewhere) one early explanation of the variable performance of portfolios, institutional or otherwise, was provided by Roll's critique of the CAPM (1977).

According to Roll, it is not only impossible for the most discerning investor to establish the composition of the *true* market portfolio, but there is also no reason to assume that a security's expected return is only affected by systematic risk. In the same year, Firth (1977) also observed that if the stock market is so efficient at assimilating all relevant information into security prices, it is impossible to claim that it is either efficient or inefficient, since by definition there is no alternative measurement criterion.

Such criticisms are important, not because they invalidate the CAPM (most empirical tests support it). But because they gave credibility to an alternative approach to portfolio asset management and security price determination based on stock market efficiency presented by Ross (1976). This is termed *Arbitrage Pricing Theory* (APT).

Unlike the CAPM, which prices securities in relation to a *global market portfolio*, the APT possesses the advantage of pricing of securities *in relation to each other*. The *single index* (beta factor) CAPM focuses upon an assumed *specific* linear relationship between betas and expected returns (systemic risk plotted by the SML). The APT is a *general* model that *subdivides* systematic risk into smaller components, which need not be specified in advance. These define the *Arbitrage Pricing Plane* (APP). Any macro-economic factors, including market sentiment, which impact upon investor returns may be incorporated into the APP (or ignored, if inconsequential.) For example, an unexpected change in the rate of inflation (purchasing power risk) might affect the price of securities generally. The advantage of the APT, however, is that it can be used to eliminate this risk specifically, such as a pension fund portfolio's requirement that it should be immune to inflation.

Statistical tests on the model, including those of Roll and Ross (1980), established that a *four factor* linear version of the APT is a more accurate predictor of security and portfolio returns than the *single factor* (index) CAPM. Specifically, their APT states that the expected return is directly proportional to its sensitivity to the following:

1) Interest rates,
2) Inflation
3) Industrial productivity,
4) Investor risk attitudes.

The return equation for a four-factor APP conforms to the following *simple linear* relationship for the expected return on the *j* th security in a portfolio:

(66) $r_j = a + b_1 (r_1) + b_2(r_2) + b_3(r_3) + b_4 (r_4)$

Where:

r_j = expected rate of return on security *j*,
r_i = expected return on factor i, (i = 1,2,3,4),
a = intercept,
b_i = slope of r_i.

The expected risk premium on the *j*th security is defined as the difference between its expected return (r_j) and the risk-free fate of interest (r_f) associated with each factor's return (r_i) and the security's sensitivity to each of these factors (b_i). The four-factor equation is given by:

$$(67)\ (r_j - r_f) = b_1\,(r_1 - r_f) + b_2\,(r_2 - r_f) + b_3\,(r_3 - r_f) + b_4\,(r_4 - r_f)$$

Like the *specific* CAPM, the *general* APT is still a *linear* model. Theoretically, it assumes that unsystematic (unique) risk can be eliminated in a well-diversified portfolio, leaving only the portfolio's sensitivity to unexpected changes in *macro-economic* factors. Subsequent studies, such as Chen, Roll and Ross (1986) therefore focused upon identifying further significant factors and why the sensitivity of returns on a particular share to each factor will vary. However, the work of Dhrymes, Friend and Gultekin (1984) had already suggested that this line of research may be redundant. Their study concluded that as the number of portfolio constituents increases, a greater number of factors must be incorporated into the model. Thus, at the limit, the APT could be equivalent to the CAPM, which defines risk in terms of a *single* over-arching *micro-economic factor* relative to the return on the market portfolio.

For one of the first comprehensive reviews of the APT, which explains why even today it is not fully developed and its application has been less successful than the CAPM, you should read Elton, Gruber and Mei (1994). A more recent perspective on the APT is provided by Huberman and Wang (2005).

8.3 Summary and Conclusions

By now you appreciate that financial analysis is not an exact science and the theories upon which it is based may even be "bad" science. The fundamental problem is that real world economic decisions are characterised by uncertainty. By definition *uncertainty is non-quantifiable*. Yet, rather than bury their heads in sand, academics continue to defend financial models, such as the CAPM based on *simplifying assumptions* that *rationalise* a search for investment opportunities in the *chaotic* world we inhabit. See Fama and French (2003).

New mathematical theories and statistical models of investor *irrationality* and market *inefficiency,* characterised by *non-random* walks are being crystallised. These *post-modern* "Quants" reject the assumptions of a normal distribution of returns. See Peters (1991) for a comprehensive exposition. Scientific "catastrophe theory" is also being applied to stock market analysis to explain why "bull" markets crash *without warning*. See Varian (2007).

Academics and financial analysts are also returning to twentieth-century economic theorists for inspiration, from John Maynard Keynes to the *behaviouralists* who dispensed with the assumption that we can maximise anything.

Today's proponents of behaviouralism, such as Montier (2002) reject the *neo-classical* economic profit motive and the wealth maximisation objectives of twentieth-century finance They believe that finance is a blend of economics and psychology that determines how investor attitudes can determine financial decisions. Explained simply, investors do not appreciate what motivates them to make one choice, rather than another. Behavioural Finance therefore seeks to explain why individuals, companies, or institutions make mistakes and how to avoid them.

Suffice it to say, that much of the "new" Quants is so complex as to confuse most financial analysts, let alone individuals who wish to beat the market (think the millennium dot.com fiasco and the 2007 meltdown). Likewise, the "new" behavioural finance (just like the "new" behavioural economics of the1960s) seems to prefer "a sledgehammer to crack a walnut" (see Hill 1990).

As a parting shot, let us therefore return to *first principles* and *common sense* with a guide to your future studies or investment plans, which places Modern Portfolio Theory in a human context.

Ignore forecasts: Evidence suggests that predictions are invariably wrong. The behavioural trait to avoid is known as *anchoring*, whereby you latch on to uncertain data that is hopelessly wrong. Develop a strategy that does not depend on them.

Information Overload: The financial services industry believes that to "beat" the market they need to know more than everybody else. But empirical studies reveal too much information leads to overconfidence, rather than accuracy. So concentrate on an investment's "key" elements.

Overconfidence: Most investors overestimate their skills. Prepare a plan based on your risk-return profile and ability. Then stick to it.

Denial: Investors are more attracted to good news, rather than bad. Prior to the millennium, the market did not want the dot.com boom to fail. So, any information suggesting that techno-shares were overvalued was ignored. The lesson is not to be complacent.

Overreaction: Investors become optimistic in a rising (bull) market and pessimistic in a falling (bear) market. When a significant proportion of investors believe that the market will rise or fall it may be a signal that the opposite will happen.

Crowd Behaviour: People feel safer herded together, which is why investors mimic the behaviour of others and buy fashionable securities and funds. Speculative investors turn this to their own advantage by acquiring stocks that are cheap and unfashionable.

Selective Memory: Most investors tend to forget failure but remember success. To beat the market and keep ahead of the crowd, keep a record of your decisions (good or bad) and learn from your mistakes.

Ignore Current Market Sentiment and Noise: Today, most investors are doing the opposite. The average holding period for a share on the New York Stock Exchange is eleven months, compared with eight years in the 1950s.

Go for long-term investment: Over time, most shareholder returns come from dividends. *But remember* the expected return from a stock is equal to the dividend yield, *plus* any dividend growth, *plus* any changes in valuation that occur. The strategy to adopt is "value investing", where you buy stocks that are cheap with high dividend yields.

To summarise:

Short-term gain equals long-term pain: According to Patrick Hosking (2010) the global financial crisis, which has cost somewhere between one and five times the entire world's financial output, started with reckless bankers lending to poor Americans. Since 2007, other contributory factors have also been suggested for the meltdown. Central banks ignored rising asset prices, governments talked up a global economic boom and financial regulators still adhered to efficient market theory by using a light touch.

However, these are merely the consequences of a more fundamental problem, motivated by greed, referred to throughout our analyses (including the *SFM* texts), namely:

> The principal-agency conflict associated with inappropriate short-term, managerial reward structures that arise from a bonus culture and lack of corporate governance, first explained by Jensen and Meckling (1976).

As Hosking observes, these flawed incentives still exist today not just in banks, but also within financial institutions and companies whose management (agents) hold shares on behalf of their owners (principles). Management rarely accept responsibility if things go wrong, but always accept rewards, even if their strategies have no lasting value. Thus, managerial short-termism rarely coincides with the long-term income and capital aspirations of their shareholders.

If proof be needed that professional portfolio management has lost its way, let us conclude with two telling UK statistics from the *London School of Economics' Centre for the Study of Capital Market Disfunctionality.*

> In real terms, pension fund returns grew by an average of 4.1 per cent per annum between 1963 and2009, but by only 1.1 per cent a year in the last 10 years.

Unless corporate management are held personally responsible for their bonuses long after their receipt (perhaps a decade) it is therefore difficult to see how the rational objectives of efficient portfolio theory can ever match the rational expectations of a portfolio's clientele.

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9 Appendix for Chapter 1

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